

# Hopf Semialgebras\*

Jawad Y. Abuhlail<sup>† ‡</sup>

Department of Mathematics and Statistics  
Box # 5046, KFUPM; 31261 Dhahran, KSA  
abuhlail@kfupm.edu.sa

Nabeela Al-Sulaiman

Department of Mathematics  
University of Dammam; 31451 Dammam, KSA  
nalsulaiman@ud.edu.sa

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## Abstract

In this paper, we introduce and investigate *bisemialgebras* and *Hopf semialgebras* over commutative semirings. We generalize to the semialgebraic context several results on bialgebras and Hopf algebras over rings including the main reconstruction theorems and the *Fundamental Theorem of Hopf Algebras*. We also provide a notion of *quantum monoids* as Hopf semialgebras which are neither commutative nor cocommutative; this extends the Hopf algebraic notion of a quantum group. The generalization to the semialgebraic context is neither trivial nor straightforward due to the non-additive nature of the base category of Abelian monoids which is also neither Puppe-exact nor homological and does not necessarily have enough injectives.

## Introduction

Topological investigations of Lie groups and group-like spaces led the German mathematician Heinz Hopf to realize that the multiplication map in the cohomology algebra yields a comultiplication, and from that combination Hopf got remarkable structure results [Hop1941]. This was the formal birth of the theory of *Hopf algebras*, one of the main streams of research in mathematics nowadays. Apart from their nice theory from the purely algebraic point of view [Swe1969], [Rad2012] (*e.g.* Kaplansky's conjectures, Andruskiewitsch-Schneider's project on the classification of semisimple finite dimensional Hopf algebras, the module theoretic approach [BW2003]), Hopf algebras play important roles in many aspect of mathematics like graded ring theory (coactions [Mon1993], [DNR2001]), algebraic geometry (affine group schemes [Abe1980], [Und2011]), number theory (formal groups), mathematical physics (quantum groups [Maj1990]), Lie algebras (universal enveloping algebras), Topology (*e.g.* cohomology of exceptional Lie groups), Knot Theory [KRT1997], non-commutative geometry,

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Galois theory (Hopf-Galois extensions), combinatorics (umbral calculus), computer science (*e.g.* models of linear logic) and many more [CGW2006].

Moreover, in category theory, *Hopf monoids* in braided monoidal categories [Tak2000] and *Hopf monads* in arbitrary categories [MW2011] are gaining increasing interest [Ver]. While some basic definitions and results remain the same when moving from one category to another [Tak1999], [CGW2006], several structural properties of the category of Hopf monoids depend naturally on the properties of the category in which such objects live (*e.g.* the category  $\mathbf{Hopf}_R$  of Hopf algebras over a commutative ring  $R$  is closed under limits in the category  $\mathbf{Bilag}_R$  of  $R$ -bialgebras if  $R$  is von Neumann regular [Por2011a]).

In this paper, we introduce and investigate *Hopf semialgebras* (*bisemialgebras*) over commutative semirings. Let  $S$  be a commutative semiring and denote by  $\mathbb{S}_S$  the category of  $S$ -semimodules and by  $\mathbb{CS}_S \xhookrightarrow{\ell} \mathbb{S}_S$  the *full* subcategory of cancellative  $S$ -semimodules [Tak1981]. A notion of Hopf semialgebras (*bisemialgebras*) was introduced by the first author<sup>1</sup> using Takahashi's tensor-like product [Tak1982], which we denote in this paper by  $\boxtimes_S$ . That notion was investigated by the second author in her dissertation [Als2011] in the category  $\mathbb{CS}_S$  assuming also that the base semiring  $S$  is *cancellative*. In this paper, we use instead the natural tensor product  $\otimes_S$  in  $\mathbb{S}_S$  inherited from the tensor product in the symmetric monoidal category  $(\mathbf{AbMonoid}, \otimes, \mathbb{N}_0)$  of Abelian monoids [Kat1997]. As clarified in [Abu-a],  $-\boxtimes_S-$  and  $-\otimes_S-$  are isomorphic bifunctors on  $\mathbb{CS}_S$ , whence the results in this paper generalize those in [Als2011]. A main advantage of using  $\otimes_S$  (instead of  $\boxtimes_S$ ) is that the category  $(\mathbb{S}_S, \otimes_S, S)$  is a symmetric monoidal category (while the category  $(S_S, \boxtimes_S, S)$  is, in general, *semiunital semimonoidal* [Abu2013]). This suggests defining Hopf semialgebras (*bisemialgebras*) as Hopf monoids (*bimonoids*) in  $(\mathbb{S}_S, \otimes_S, S)$ .

Such Hopf monoids (*bimonoids*) not only add *new families of concrete examples* to the literature, but they are of particular importance for theoretical and practical reasons. On one hand, and in contrast to the category of modules over a ring, the category of semimodules over a semiring is not Abelian (not even additive) and so many proofs that depend heavily on lemmas of diagrams cannot be directly applied to our context. Add to that this category is not *Puppe-exact* and not *homological* [BB2004] and so a new notion of exact sequences for semimodules over a semiring was necessary to prove restricted versions of the Short Five Lemma and the Snake Lemma [Abu-b]. Moreover, working over proper semifields (semisimple proper semirings) does not bring big advantages as was the case in the theory of Hopf algebras over fields (semisimple rings). This is due to the fact that all semivector spaces over a semifield  $F$  (semimodules over a semisimple semiring  $S$ ) are free (projective) if and only if  $F$  is a field [KN2011, Theorem 5.11] ( $S$  is a semisimple ring [KN2011, Theorem 5.7]). This suggests that one uses a combination of techniques from categorical algebra, universal algebra and homological algebra to overcome these and other difficulties [Abu-c]. On the other hand, semirings and semimodules proved to have a wide spectrum of significant applications in several aspects of mathematics like optimization theory, tropical geometry, idempotent analysis, physics, theoretical computer science (*e.g.* Automata Theory) and many more [Gol1999]. It is hoped that investigating Hopf semialgebras would bring new applications (for some applications of bisemialgebras in Automata Theory, see [Wor2012]).

This paper is organized as follows: after this introduction, we recall in Section 1 some basic definitions and properties of semicoalgebras and semicomodules (for a detailed discussion of semicorings and semicomodules, see [Abu-c]). In Section 2, we introduce the notion of a bisemialgebra and study *integrals on* and *in* a given bisemialgebra. Moreover, we give a recon-

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<sup>1</sup><http://www.ingvet.kau.se/juerfuch/conf/nomap/talk/Abuhlail.pdf>

struction result for bisemialgebras using the notion of a *bimonad* in the sense of [BBW2009] (Theorem 2.32). In Section 3, we investigate the categories of Doi-Koppinen semimodules and give relatively weak sufficient conditions for such a category to be a Wisbauer category of type  $\sigma[M]$  for a suitable *subgenerator*  $M$  (Theorem 3.9). In Section 4, we consider Hopf semialgebras and extend several examples of quantum groups to *quantum monoids* which we introduce as non-commutative non-cocommutative Hopf semialgebras. Moreover, we present the *Fundamental Theorem of Hopf Semialgebras* (Theorem 4.14). A reconstruction result for Hopf semialgebras in terms of Hopf monads [MW2011] is also obtained (Theorem 4.16). In addition to that, we use integrals to characterize Hopf semialgebras which are cosemisimple as semicoalgebras (Proposition 4.23) and those which are semisimple as semialgebras (Proposition 4.26). In Section 5, we present possible constructions of *dual* semicoalgebras, dual bisemialgebras and dual Hopf semialgebras in both the finite and the infinite cases.

## 1 Preliminaries

In this section, we collect some definitions and properties of semirings (semimodules) and semicoalgebras (semicomodules).

### Semirings and Semimodules

A *semiring* is essentially a *monoid* in the category **AbMonoid** of Abelian Monoids, or – roughly speaking – a ring not necessarily with subtraction. Moreover, a *semifield* is a semiring in which every non-zero element is invertible. Indeed, any ring is a semiring; the first natural examples of semirings (semifields) which are not rings (not fields) are the set  $\mathbb{N}_0$  of natural numbers ( $\mathbb{R}^+ := [0, \infty)$  and  $\mathbb{Q}^+ := \mathbb{Q} \cap [0, \infty)$ ) with the usual addition and multiplication. All semirings in this paper are unital, and for a given semiring  $S$ , we assume that  $0_S \neq 1_S$ . Given a semiring  $S$ , we mean by a right (left)  $S$ -semimodule a right (left)  $S$ -module not necessarily with subtraction. The category of right (left)  $S$ -semimodules is denoted by  $\mathbb{S}_S$  ( ${}_S\mathbb{S}$ ). For two semirings  $S$  and  $T$ , an  $(S, T)$ -bisemimodule is a left  $S$ -semimodule, which is also a right  $T$ -semimodule such that  $(sm)t = s(mt)$  for all  $s \in S$ ,  $m \in M$  and  $t \in T$ . The category of  $(S, T)$ -bisemimodules and  $S$ -linear  $T$ -linear maps (called  $(S, T)$ -*bilinear maps*) is denoted by  ${}_S\mathbb{S}_T$ . We refer the reader to [Gol1999] and the first section of [Abu-c] for the basic definitions and properties of semirings and semimodules. For any right (left)  $S$ -semimodule, we have a canonical isomorphism of Abelian monoids  $M \otimes_S S \xrightarrow{\vartheta_M^r} M$  ( $S \otimes_S M \xrightarrow{\vartheta_M^l} M$ ).

Before we proceed, we give a number of examples of members in an important family of semirings which does not include any (non-zero) ring, *i.e.* every semiring in this family is *proper*.

**Definition 1.1.** A semiring  $(S, +_S, \cdot_S, \mathbf{0}, \mathbf{1})$  is said to be *additively idempotent* iff  $\mathbf{1} + \mathbf{1} = \mathbf{1}$ , or equivalently iff  $a + a = a$  for every  $a \in S$ .

*Remark 1.2.* If  $(S, +_S, \cdot_S, \mathbf{0}, \mathbf{1})$  is an additively idempotent semiring, then

$$\mathbf{n} := \underbrace{\mathbf{1} +_S \cdots +_S \mathbf{1}}_{n \text{ times}} = \mathbf{1} = \mathbf{1} \cdot_S \mathbf{1} \neq \mathbf{0}.$$

*Example 1.3.* Every distributive complete lattice  $\mathcal{L} = (L, \vee, \wedge, \mathbf{0}, \mathbf{1})$  is an additively idempotent semiring:  $\mathbf{1} +_{\mathcal{L}} \mathbf{1} = \mathbf{1} \vee \mathbf{1} = \mathbf{1}$ . In particular, for any ring  $R$ , the lattice  $\mathcal{L}(R) :=$

(Ideal( $R$ ),  $+$ ,  $\cdot$ ,  $R$ ,  $0$ ) of (two-sided) ideals of  $R$  is distributive and complete whence an additively idempotent semiring:  $\mathbf{1} +_{\mathcal{L}} \mathbf{1} = R + R = R = \mathbf{1}$ . Notice that  $\mathcal{L}(R)$  has no non-zero zerodivisors if and only if  $\{0_R\}$  is a prime ideal (*i.e.*  $R$  is a prime ring).

*Example 1.4.*  $\mathbb{B} = \{0, 1\}$  is an additively idempotent semiring with addition:  $0 + 0 = 0$  and  $1 + 1 = 1$  (called the *Boolean semifield*). Notice that  $\mathbb{B} \not\cong \mathbb{Z}/2\mathbb{Z}$ . The semiring  $\mathbb{B}$  has many applications in automata theory and in switching theory where it is called the *switching algebra* ([Gol1999, p. 7]).

**Proposition 1.5.** *Let  $S$  be a semiring.*

1.  $\mathbb{S}_S$  is complete and cocomplete. In particular, it has equalizers (kernels) and coequalizers (cokernels).
2.  $S_S$  is a regular generator in  $\mathbb{S}_S$  (in the sense of [BW2005, p. 199]).
3. If  $S$  is commutative, then  $(\mathbb{S}_S, \otimes_S, S; \tau)$  is a symmetric monoidal category with symmetric braiding

$$\tau_{(M,N)} : M \otimes_S N \simeq N \otimes_S M, \quad m \otimes_S n \mapsto n \otimes_S m.$$

The proofs of the following lemmata are similar to those for modules over a ring (*e.g.* [Wis1991, 12.9, 25.5 (2)]) by applying a relaxed version of the Short Five Lemma for semi-modules over semirings [Abu-a, Lemma 1.22].■

**Definition 1.6.** Let  $M$  and  $N$  be  $S$ -semimodules. We call an  $S$ -linear map  $f : M \longrightarrow N$  : *i-uniform* (*image-uniform*) iff

$$f(M) = \overline{f(M)} := \{n \in N \mid n + f(m) = f(m') \text{ for some } m, m' \in M\};$$

*k-uniform* (*kernel-uniform*) iff for all  $m, m' \in M$  we have

$$f(m) = f(m') \Rightarrow m + k = m' + k' \text{ for some } k, k' \in \text{Ker}(f); \quad (1)$$

*uniform* iff  $f$  is *i-uniform* and *k-uniform*.

We call  $L \leq_S M$  a *uniform subsemimodule* iff the embedding  $L \xrightarrow{\iota_L} M$  is (*i*-)uniform, or equivalently iff  $L \leq_S M$  is subtractive. If  $\equiv$  is an  $S$ -congruence on  $M$  [Gol1999], then we call  $M/\equiv$  a *uniform quotient* iff the projection  $\pi_{\equiv} : M \longrightarrow M/\equiv$  is (*k*-)uniform.

**Definition 1.7.** We say that an  $S$ -semimodule  $X$  (*uniformly*) *generates*  $M_S$  iff there exists an index set  $\Lambda$  and a (uniform) surjective  $S$ -linear map  $X^{(\Lambda)} \xrightarrow{\pi} M \longrightarrow 0$ . With  $\text{Gen}(X)$  we denote the class of  $S$ -semimodules generated by  $X_S$ .

**Definition 1.8.** We say that  $X_S$  is

*uniformly* (finitely) *generated* iff there exists a (finite) index set  $\Lambda$  and a uniform surjective  $S$ -linear map  $S^{(\Lambda)} \longrightarrow X \longrightarrow 0$ ;

*finitely presented* iff  $\text{Hom}_S(X, -) : S_S \longrightarrow \mathbf{AbMonoid}$  preserves directed colimits (*i.e.*  $X \in \mathbb{S}_S$  is a finitely presentable object in the sense of [AP1994]);

*uniformly finitely presented* iff  $X$  is uniformly finitely generated and for any exact sequence of  $S$ -semimodules

$$0 \longrightarrow K \xrightarrow{f} S^n \xrightarrow{g} X \longrightarrow 0,$$

the  $S$ -semimodule  $K$  ( $\simeq \text{Ker}(g)$ ) is finitely generated.

**1.9.** Let  $M$  be a right  $S$ -semimodule. With  $\sigma[M_S]$  ( $\sigma_u[M_S]$ ) we denote the closure of  $\text{Gen}(M_S)$  under (*uniform*)  $S$ -subsemimodules, *i.e.* the smallest full subcategory of  $\mathbb{S}_S$  which contains  $M_S$  and is closed under direct sums, homomorphic images and (*uniform*)  $S$ -subsemimodules. We say that  $M_S$  is a (*uniformly*) *subgenerator* for  $\sigma[M_S]$  ( $\sigma_u[M_S]$ ). Notice that  $\text{Gen}(M_S) \subseteq \sigma_u[M_S] \subseteq \sigma[M_S]$ .

**Lemma 1.10.** *Let  $M_S$  be a right  $S$ -semimodule,  $\{L_\lambda\}_\Lambda$  a class of left  $S$ -semimodules and consider the canonical map*

$$\varphi_M : M \otimes_S \prod_{\lambda \in \Lambda} L_\lambda \longrightarrow \prod_{\lambda \in \Lambda} (M \otimes_S L_\lambda), \quad m \otimes_S (l_\lambda)_\Lambda \mapsto (m \otimes_S l_\lambda)_\Lambda. \quad (2)$$

1.  $M_S$  is finitely generated if and only if  $\varphi_M$  is surjective.
2. If  $M_S$  is uniformly finitely presented, then  $\varphi_M$  is an isomorphism.

**Definition 1.11.** ([Kat2004], [Abu-a]) We call a right  $S$ -semimodule  $M$  :

*flat* iff  $M \otimes_A -$  is left exact, *i.e.* it preserves finite limits, equivalently  $M \simeq \varinjlim F_\lambda$ , a filtered limit of finitely generated free right  $S$ -semimodules;

*uniformly flat* iff  $M \otimes_A - : {}_A\mathbb{S} \longrightarrow \mathbf{AbMonoid}$  preserves *uniform* subobjects;

*mono-flat* iff  $M \otimes_A - : {}_A\mathbb{S} \longrightarrow \mathbf{AbMonoid}$  preserves monomorphisms (injective  $S$ -linear maps);

*u-flat* iff  $M \otimes_A - : {}_A\mathbb{S} \longrightarrow \mathbf{AbMonoid}$  sends (*uniform*) monomorphisms to (*uniform*) monomorphisms;

*projective* iff  $M$  is a retract of a free  $S$ -semimodule, or equivalently, iff  $M$  has a *dual basis*.

**Lemma 1.12.** *Let  $S, T$  be semirings,  $L$  a right  $S$ -semimodule and  $K$  a  $(T, S)$ -bisemimodule. Let  $Q_T$  be a right  $T$ -semimodule and consider the canonical morphism*

$$v_{(Q,L,K)} : Q \otimes_T \text{Hom}_S(L, K) \longrightarrow \text{Hom}_S(L, Q \otimes_T K), \quad q \otimes_T h \mapsto [l \mapsto q \otimes h(l)].$$

1. If  $Q_T$  is mono-flat and  $L_S$  is finitely generated, then  $v_{(Q,L,K)}$  is injective.
2. If  $Q_T$  is uniformly flat and  $L_S$  is uniformly finitely presented, then  $v_{(Q,L,K)}$  is surjective.
3. If  $Q_T$  is flat and  $L_S$  is uniformly finitely presented, then  $v_{(Q,L,K)}$  is an isomorphism.

## Semicoalgebras and Semicomodules

Throughout,  $S$  is a commutative semiring with  $1_S \neq 0_S$ .

**1.13.** An  *$S$ -semialgebra* is a triple  $(A, \mu_A, \eta_A)$  where  $A$  is an  $S$ -semimodule and  $\mu_A : A \otimes_S A \longrightarrow A$ ,  $\eta_A : S \longrightarrow A$  are  $S$ -linear maps such that the following diagrams are commutative

$$\begin{array}{ccc} A \otimes_S A \otimes_S A & \xrightarrow{\mu_A \otimes_S A} & A \otimes_S A \\ \downarrow A \otimes_S \mu_A & & \downarrow \mu_A \\ A \otimes_S A & \xrightarrow{\mu_A} & A \end{array} \quad \begin{array}{ccccc} S \otimes_S A & \xrightarrow{\eta_A \otimes_S A} & A \otimes_S A & \xleftarrow{A \otimes_S \eta_A} & A \otimes_S S \\ & \searrow \vartheta_A^l & \downarrow \mu_A & \swarrow \vartheta_A^r & \\ & & A & & \end{array}$$

We call  $\mu_A$  the *multiplication* and  $\eta_A$  the *unity* of  $A$ . Let  $A$  and  $B$  be  $S$ -semialgebras. We call an  $S$ -linear map  $f : A \rightarrow B$  an  *$S$ -semialgebra morphism* iff the following diagrams are commutative

$$\begin{array}{ccc} A \otimes_S A & \xrightarrow{f \otimes_S f} & B \otimes_S B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta_A \swarrow & & \searrow \eta_B \\ & S & \end{array}$$

The set of morphisms of  $S$ -semialgebras from  $A$  to  $B$  is denoted by  $\mathbf{SAlg}_S(A, B)$ . The category of  $S$ -semialgebras will be denoted by  $\mathbf{SAlg}_S$ .

Semicoalgebras are dual to semialgebras and are defined by reversing the arrows in the diagrams mentioned above.

**1.14.** An  *$S$ -semicoalgebra* is a triple  $(C, \Delta_C, \varepsilon_C)$  in which  $C$  is an  $S$ -semimodule and  $\Delta_C : C \rightarrow C \otimes_S C$ ,  $\varepsilon_C : C \rightarrow S$  are  $S$ -linear maps such that the following diagrams are commutative

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes_S C \\ \Delta_C \downarrow & & \downarrow C \otimes_S \Delta_C \\ C \otimes_S C & \xrightarrow{\Delta_C \otimes_S C} & C \otimes_S C \otimes_S C \end{array} \quad \begin{array}{ccccc} & & C & & \\ \vartheta_C^l \nearrow & & \downarrow \Delta_C & & \nwarrow \vartheta_C^r \\ S \otimes_S C & \xleftarrow{\varepsilon_C \otimes_S C} & C \otimes_S C & \xrightarrow{C \otimes_S \varepsilon_C} & C \otimes_S S \end{array}$$

We call  $\Delta_C$  the *comultiplication* and  $\varepsilon_C$  the *counity* of  $C$ . For  $S$ -semicoalgebras  $C$  and  $D$ , we call an  $S$ -linear map  $f : D \rightarrow C$  an  *$S$ -semicoalgebra morphism* iff the following diagrams are commutative

$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ \Delta_D \downarrow & & \downarrow \Delta_C \\ D \otimes_S D & \xrightarrow{f \otimes_S f} & C \otimes_S C \end{array} \quad \begin{array}{ccc} D & \xrightarrow{f} & C \\ \varepsilon_D \swarrow & & \searrow \varepsilon_C \\ & S & \end{array}$$

The set of  $S$ -semicoalgebra morphisms from  $D$  to  $C$  is denoted by  $\mathbf{SCoal}_S(D, C)$ . The category of  $S$ -semicoalgebras is denoted by  $\mathbf{SCoal}_S$ .

**Notation.** Let  $(C, \Delta, \varepsilon)$  be an  $S$ -semicoalgebra. We use Sweedler-Heyneman's  $\sum$ -notation, and write for  $c \in C$  :

$$\begin{aligned} \Delta(c) &= \sum c_1 \otimes_S c_2 \in C \otimes_S C; \\ \sum c_{11} \otimes_S c_{12} \otimes_S c_2 &= \sum c_1 \otimes_S c_2 \otimes_S c_3 = \sum c_1 \otimes_S c_{21} \otimes_S c_{22}. \end{aligned}$$

**1.15.** Notice that an  $S$ -semialgebra  $(A, \mu, \eta)$  is *commutative* iff  $\mu_A \circ \tau_{(A,A)} = \mu_A$ ; with  ${}_{\mathbf{c}}\mathbf{SAlg}_S \hookrightarrow \mathbf{SAlg}_S$ , we denote the category of commutative  $S$ -semialgebras. Dually, an  $S$ -semicoalgebra  $(C, \Delta, \varepsilon)$  is said to be *cocommutative* iff  $\tau_{(C,C)} \circ \Delta = \Delta$ , i.e.  $\sum c_1 \otimes_S c_2 = \sum c_2 \otimes_S c_1$  for all  $c \in C$ . With  ${}_{\text{coc}}\mathbf{SCoal}_S \hookrightarrow \mathbf{SCoal}_S$ , we denote the *full* subcategory of cocommutative  $S$ -semicoalgebras.

*Example 1.16.* Let  $M$  be an  $S$ -semimodule. We have an  $S$ -semicoalgebra structure on  $C = (S \oplus M, \Delta, \varepsilon)$ , where

$$\begin{aligned} \Delta &: (s, m) \mapsto (s, 0) \otimes_S (1, 0) + (1, 0) \otimes_S (0, m) + (0, m) \otimes_S (1, 0); \\ \varepsilon &: (s, m) \mapsto s. \end{aligned}$$

Notice that there are many properties  $\mathbb{P}$  such that  ${}_S C$  has Property  $\mathbb{P}$  if (and only if)  ${}_S M$  has Property  $\mathbb{P}$ , *e.g.* being flat, (finitely) projective, finitely generated [Wisch1975, Example 10 (1)].

*Example 1.17.* Let  $X$  be any set. We have an  $S$ -semicoalgebra  $(S[X], \Delta, \varepsilon)$ , where  $S[X]$  is the free  $S$ -semimodule with basis  $X$  and  $\Delta, \varepsilon$  are defined by extending the following assignments linearly

$$\Delta : S[X] \mapsto S[X] \otimes_S S[X], \quad x \mapsto x \otimes_S x \text{ and } \varepsilon : S[X] \mapsto S, \quad x \mapsto 1_S.$$

## Semicomodules

Dual to semimodules of semialgebras are semicomodules of semicoalgebras:

**1.18.** Let  $(C, \Delta, \varepsilon)$  be an  $S$ -semicoalgebra. A right  $C$ -semicomodule is an  $S$ -semimodule  $M$  associated with an  $S$ -linear map (called  $C$ -coaction)

$$\rho^M : M \longrightarrow M \otimes_S C, \quad m \mapsto \sum m_{<0>} \otimes_S m_{<1>},$$

such that the following diagrams are commutative

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M \otimes_S C \\ \rho^M \downarrow & & \downarrow M \otimes_S \Delta \\ M \otimes_S C & \xrightarrow{\rho^M \otimes_S C} & M \otimes_S C \otimes_S C \end{array} \quad \begin{array}{ccc} M & \xrightarrow{\rho^M} & M \otimes_S C \\ \vartheta_M^r \swarrow & & \swarrow M \otimes_S \varepsilon \\ & M \otimes_S S & \end{array}$$

Let  $M$  and  $N$  be right  $C$ -semicomodules. We call an  $S$ -linear map  $f : M \longrightarrow N$  a  $C$ -semicomodule morphism (or  $C$ -colinear) iff the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \rho^M \downarrow & & \downarrow \rho^N \\ M \otimes_S C & \xrightarrow{f \otimes_S C} & N \otimes_S C \end{array}$$

The set of  $C$ -colinear maps from  $M$  to  $N$  is denoted by  $\text{Hom}^C(M, N)$ . The category of right  $C$ -semicomodules and  $C$ -colinear maps is denoted by  $\mathbb{S}^C$ . For a right  $C$ -semicomodule  $M$ , we call  $L \leq_A M$  a  $C$ -subsemicomodule iff  $(L, \rho^L) \in \mathbb{S}^C$  and the embedding  $L \xrightarrow{\iota_L} M$  is  $C$ -colinear. Symmetrically, we define the category  ${}^C\mathbb{S}$  of left  $C$ -semicomodules. For two left  $C$ -semicomodules  $M$  and  $N$ , we denote by  ${}^C\text{Hom}(M, N)$  the set of  $C$ -colinear maps from  $M$  to  $N$ .

**1.19.** Let  $(M, \rho^{(M;C)})$  be a right  $C$ -semicomodule,  $(M, \rho^{(M;D)})$  a left  $D$ -semicomodule and consider the left  $D$ -semicomodule  $(M \otimes_S C, \rho^{(M;D)} \otimes_S C)$  (the right  $C$ -semicomodule  $(D \otimes_S M, D \otimes_S \rho^{(M;C)})$ ). We call  $M$  a  $(D, C)$ -bisemicomodule iff  $\rho^{(M;C)} : M \longrightarrow M \otimes_S C$  is  $D$ -colinear, or equivalently iff  $\rho^{(M;D)} : M \longrightarrow D \otimes_S M$  is  $C$ -colinear. For  $(D, C)$ -bisemicomodules  $M$  and  $N$ , we call a  $D$ -colinear  $C$ -colinear map  $f : M \longrightarrow N$  a  $(D, C)$ -bisemicomodule morphism (or  $(D, C)$ -bilinear). The category of  $(D, C)$ -bisemicomodules and  $(D, C)$ -bilinear maps is denoted by  ${}^D\mathbb{S}^C$ .

*Remark 1.20.* Let  $(C, \Delta, \varepsilon)$  be an  $S$ -semicoalgebra. If  $(M, \rho^M)$  is a right  $C$ -semicomodule, then  $\rho^M : M \longrightarrow M \otimes_S C$  is a splitting monomorphism in  $\mathbb{S}_A$ ; however,  $M$  is not necessarily a direct summand of  $M \otimes_S C$ ; see [Gol1999, 16.6].

*Example 1.21.* Let  $M = \bigoplus_{g \in G} M_g$  be a  $G$ -graded  $S$ -semimodules, where  $G$  is group. One can consider  $M$  as an  $S[G]$ -semimodule with

$$\rho^M : M \longrightarrow M \otimes_S S[G], \quad \sum_{g \in G} m_g \mapsto \sum_{g \in G} m_g \otimes_S g.$$

Conversely, if  $M$  is an  $S[G]$ -semicomodule, then  $M$  is a  $G$ -graded  $S$ -semimodule in the canonical way. In fact, we have an isomorphism of categories  $\mathbf{gr}_G(\mathbb{S}_S) \simeq \mathbb{S}^{S[G]}$ , where  $\mathbf{gr}_G(\mathbb{S}_S)$  is the category of  $G$ -graded  $S$ -semimodules.

**1.22.** We have an isomorphism of categories  $\mathbf{SAlg}_S \simeq \mathbf{Monoid}(\mathbb{S}_S)$ . An  $S$ -semialgebra  $A$  is essentially a *monoid* in  $\mathbb{S}_S$  and so it induces two *monads*  $- \otimes_S A : \mathbb{S}_S \longrightarrow \mathbb{S}_S$  and  $A \otimes_S - : \mathbb{S}_S \longrightarrow \mathbb{S}_S$ . Moreover, we have isomorphisms of categories

$$\mathbb{S}_A \simeq (\mathbb{S}_S)_{-\otimes_S A} \text{ and } {}_A\mathbb{S} \simeq (\mathbb{S}_S)_{A\otimes_S -}.$$

We have an isomorphism of categories  $\mathbf{SAlg}_S \simeq \mathbf{Comonoid}(\mathbb{S}_S)$ . An  $S$ -semicoalgebra  $C$  is essentially a *comonoid* in  $\mathbb{S}_S$  and so it induces two *comonads*  $- \otimes_S C : \mathbb{S}_S \longrightarrow \mathbb{S}_S$  and  $C \otimes_S - : \mathbb{S}_S \longrightarrow \mathbb{S}_S$ . Moreover, we have isomorphisms of categories

$$\mathbb{S}^C \simeq \mathbb{S}_S^{-\otimes_S C} \text{ and } {}^C\mathbb{S} \simeq \mathbb{S}_S^{C\otimes_S -}.$$

**1.23.** Let  $C$  be an  $S$ -semicoalgebra. For every  $S$ -semialgebras  $A$ , there is a canonical structure of an  $S$ -semialgebra on  $\text{Hom}_S(C, A)$  with multiplication given by the *convolution product*

$$(f * g)(c) = \sum f(c_1)g(c_2) \text{ for all } f, g \in \text{Hom}_S(C, A) \text{ and } c \in C. \quad (3)$$

In particular,  $C^* := \text{Hom}_S(C, S)$  is an  $S$ -semialgebra and  $C$  is a  $(C^*, C^*)$ -bisemimodule with left and right actions given by

$$f \rightharpoonup c := \sum c_1 f(c_2) \text{ and } c \leftharpoonup g := \sum f(c_1) c_2 \text{ for all } f, g \in C^* \text{ and } c \in C.$$

**1.24.** Let  $C$  be an  $S$ -semicoalgebra. We say that  ${}_S C$  is an  $\alpha$ -semimodule (or  ${}_S C$  satisfies the  $\alpha$ -condition) iff for every  $M_S$ , the canonical map

$$\alpha_M^C : M \otimes_S C \longrightarrow \text{Hom}_S(C^*, M), \quad m \otimes_S c \mapsto [f \mapsto mf(c)]$$

is injective and uniform. Clearly, every right  $C$ -semicomodule  $M$  is a left  ${}^*C$ -semimodule with

$$f \rightharpoonup m := \sum m_{<0>} f(m_{<1>}) \text{ for all } f \in C^* \text{ and } m \in M.$$

If  ${}_S C$  is an  $\alpha$ -semimodule, then for every  ${}^*C M$  with induced map  $\tilde{\rho}_M : M \longrightarrow \text{Hom}_S(C^*, M)$ , we define the  $C$ -rational subsemimodule of  $M$  as  $\text{Rat}^C({}^*C M) := \tilde{\rho}_M^{-1}(\alpha_M^C(M \otimes_S C))$ . In this case, we have by [Abu-c, Theorem 3.16] an isomorphism of categories

$$\mathbb{S}^C \simeq \text{Rat}^C({}^*C \mathbb{S}). \quad (4)$$



## 2 Bisemialgebras

**2.1.** With an  $S$ -bisemialgebra, we mean a datum  $(B, \mu, \eta, \Delta, \varepsilon)$ , where  $(B, \mu, \eta)$  is an  $S$ -semialgebra and  $(B, \Delta, \varepsilon)$  is an  $S$ -semicoalgebra such that  $\Delta : B \rightarrow B \otimes_S B$  and  $\varepsilon : B \rightarrow S$  are morphisms of  $S$ -semialgebras, or equivalently  $\mu : B \otimes_S B \rightarrow B$  and  $\eta : S \rightarrow B$  are morphisms of  $S$ -semicoalgebras; notice that  $B \otimes_S B$  can be given a structure of a  $S$ -semialgebra ( $S$ -semicoalgebra) in a canonical way using the twisting map  $\tau_{(B,B)}$ . A *morphism of  $S$ -bisemialgebras*  $f : B \rightarrow B'$  is an  $S$ -linear map which is simultaneously a morphism of  $S$ -semialgebras and a morphism of  $S$ -semicoalgebras. The category of  $S$ -bisemialgebras is denoted by  $\mathbf{SBiAlg}_S$ .

**Notation.** Given an  $S$ -bisemialgebra  $B$ , we write  $B^a$  when we handle  $B$  as an  $S$ -semialgebra and  $B^c$  when we consider  $B$  as an  $S$ -semicoalgebra.

*Example 2.2.*  $S$  is an  $S$ -bisemialgebra with

$$\begin{aligned}\Delta_S &: S \mapsto S \otimes_S S, s \mapsto s \otimes_S 1_S = 1_S \otimes_S s; \\ \varepsilon_S &: S \rightarrow S, s \mapsto s.\end{aligned}$$

*Example 2.3.* If  $B$  is an  $S$ -bisemialgebra, then  $S$  is a  $(B, B)$ -bisemicomodule with

$${}^S\rho : S \rightarrow B \otimes_S S, s \mapsto 1_B \otimes_S s \text{ and } \rho^S : S \rightarrow S \otimes_S B, s \mapsto s \otimes_S 1_B.$$

*Example 2.4.* Let  $A$  be an  $S$ -semialgebra and consider  $B = S \oplus A$  as an  $S$ -semialgebra with point wise multiplication and unity  $(1_S, 1_A)$ . It is obvious that  $B$  has a structure of an  $S$ -bisemialgebra with

$$\begin{aligned}\Delta &: B \rightarrow B \otimes_S B, (s, a) \mapsto (s, 0) \otimes_S (1_S, 0) + (1_S, 0) \otimes_S (0, a) + (0, a) \otimes_S (1_S, 0); \\ \varepsilon &: B \rightarrow S, (s, a) \mapsto s.\end{aligned}$$

*Example 2.5.* Let  $(G, *, e)$  be a monoid and consider the free  $S$ -semimodule  $S[G]$  as an  $S$ -semialgebra with multiplication induced by  $*$  and unity  $1 = 1_S e$ . One can easily see that  $S[G]$  has two  $S$ -semicoalgebra structures which are compatible with the  $S$ -semialgebra structure yielding two  $S$ -bisemialgebra structures  $(S[G], *, 1, \Delta, \varepsilon)$  and  $(S[G], *, 1, \tilde{\Delta}, \tilde{\varepsilon})$  where the comultiplications and the counities are obtained by extending the following assignments as  $S$ -semialgebra morphisms

$$\begin{aligned}\Delta &: S[G] \rightarrow S[G] \otimes_S S[G], g \mapsto g \otimes_S g \text{ and } \varepsilon : S[G] \rightarrow S, g \mapsto 1_S; \\ \tilde{\Delta} &: S[G] \rightarrow S[G] \otimes_S S[G], g \mapsto g \otimes_S 1 + 1 \otimes_S g \text{ and } \tilde{\varepsilon} : S[G] \rightarrow S, g \mapsto \delta_{e,g}.\end{aligned}$$

*Example 2.6.* The previous example applies in particular to the polynomial  $S$ -semialgebra  $S[x]$  since we have an isomorphism of monoids  $M := (\{1, X, \dots, X^n, \dots\}, \cdot) \simeq (\mathbb{N}_0, +)$ , whence  $S[x]$  has two  $S$ -bisemialgebra structures  $(S[x], \cdot, 1, \Delta_1, \varepsilon_1)$  and  $(S[x], \cdot, 1, \Delta_2, \varepsilon_2)$  with

$$\begin{aligned}\Delta &: S[x] \rightarrow S[x] \otimes_S S[x], \sum_{i=0}^n s_i x^i \mapsto \sum_{i=0}^n s_i x^i \otimes_S x^i; \\ \varepsilon &: S[x] \rightarrow S, \sum_{i=0}^n s_i x^i \mapsto \sum_{i=0}^n s_i;\end{aligned}$$

and

$$\begin{aligned}\tilde{\Delta} & : S[x] \longrightarrow S[x] \otimes_S S[x], \quad \sum_{i=0}^n s_i x^i \mapsto \sum_{i=0}^n s_i \left( \sum_{j=0}^i \binom{i}{j} x^j \otimes_S x^{i-j} \right); \\ \tilde{\varepsilon} & : S[x] \longrightarrow S, \quad \sum_{i=0}^n s_i x^i \mapsto s_0.\end{aligned}$$

*Example 2.7.* (cf. [Wor2012]) Consider the Boolean semiring  $\mathbf{B} = \{0, 1\}$ . Let  $P = \mathbf{B} \langle x, y \mid xy \neq yx \rangle$  be the  $\mathbf{B}$ -semimodule of formal sums of words formed from the *non-commuting* letters  $x$  and  $y$ . In fact,  $P$  is a non-commutative  $\mathbf{B}$ -semialgebra with multiplication given by *concatenation* of words (e.g.  $(xyxx) \cdot (yyx) = xyxxyyx$ ) and unity  $\square$  (the empty word). It can be easily seen that the structure maps of the following  $\mathbf{B}$ -semicoalgebras can be extended as  $S$ -semialgebra morphisms yielding two different  $\mathbf{B}$ -bisemialgebra structures on  $P$ :

1.  $(P, \Delta, \varepsilon)$ , where

$$\Delta_1 : P \longrightarrow P \otimes_{\mathbf{B}} P, \quad w \mapsto w \otimes_{\mathbf{B}} w \text{ and } \varepsilon_1 : P \longrightarrow \mathbf{B}, \quad w \mapsto w(1, 1).$$

2.  $(P, \tilde{\Delta}, \tilde{\varepsilon})$ , where

$$\begin{aligned}\tilde{\Delta} & : P \longrightarrow P \otimes_{\mathbf{B}} P, \quad x \mapsto x \otimes_{\mathbf{B}} \square + \square \otimes_{\mathbf{B}} x, \quad \Delta(y) = y \otimes_{\mathbf{B}} \square + \square \otimes_{\mathbf{B}} y; \\ \tilde{\varepsilon} & : P \longrightarrow \mathbf{B}, \quad w \mapsto w(0, 0) \text{ (notice that } \square(0, 0) = 1_S \text{ by convention).}\end{aligned}$$

*Example 2.8.* (cf. [HGK2010, 3.2.20], [Wor2012]) Let  $S$  be a semiring and consider the free  $S$ -semimodule  $B = S \langle \mathbb{N} \rangle$  with basis all *words* on  $\mathbb{N}$ . Notice that  $B$  is an  $S$ -semialgebra with multiplication given by the *concatenation* of words and unity  $\square$ , the empty word. Moreover,  $B$  is an  $S$ -semicoalgebra with comultiplication and counity given by

$$\Delta : B \longrightarrow B \otimes_S B, \quad w \mapsto \sum_{w_1 w_2 = w} w_1 \otimes_S w_2 \text{ and } \varepsilon : B \longrightarrow S, \quad w \mapsto w(0, \dots, 0).$$

However, these  $S$ -semialgebra and  $S$ -semicoalgebra structures are – in general – *not* compatible and so do not yield a structure of an  $S$ -bisemialgebra on  $B$ ; for example, we have

$$\begin{aligned}\Delta([2]) \cdot \Delta([3]) & = (\square \otimes_R [2] + [2] \otimes_R \square) \cdot (\square \otimes_R [3] + [3] \otimes_R \square) \\ & = \square \otimes_R [2, 3] + [2] \otimes_R [3] + [3] \otimes_R [2] + [2, 3] \otimes_R \square,\end{aligned}$$

while

$$\Delta([2, 3]) = \square \otimes_R [2, 3] + [2] \otimes_R [3] + [2, 3] \otimes_R \square.$$

*Example 2.9.* (cf. [Str2007, Example 7.9]) Let the commutative semiring  $S$  be additively idempotent,  $E = \{e_0, e_1, e_2, \dots, e_n, \dots\}$  a countable set,  $B := S[E]$  the free  $S$ -semimodule and consider the assignments

$$\begin{aligned}\mu_E & : B \otimes_S B \longrightarrow B, \quad e_p \otimes_S e_q \mapsto e_{p+q}; \\ \eta & : S \longrightarrow B, \quad 1_S \mapsto e_0; \\ \Delta & : B \longrightarrow B \otimes_S B, \quad e_n \mapsto \sum_{p+q=n} e_p \otimes_S e_q; \\ \varepsilon & : B \longrightarrow S, \quad e_n \mapsto \delta_{0,n}.\end{aligned}$$

Extending these assignments linearly such that  $\mu$  and  $\eta$  are  $S$ -semialgebra morphisms, we obtain a structure of an  $S$ -bisemialgebra on  $B$ .

*Example 2.10.* ([BW2003, 15.12]) Associated to every  $S$ -semimodule  $M$  is the so-called  $S$ -tensor-semialgebra

$$\mathcal{T}(M) = (S \oplus M \oplus (M \otimes_S M) \oplus (M \otimes_S M \otimes_S M) \oplus \cdots, \mu, \eta)$$

where the multiplication and the unity are given by

$$\begin{aligned} \mu((m_1 \otimes_S \cdots \otimes_S m_n)(m'_1 \otimes_S \cdots \otimes_S m'_t)) &: = m_1 \otimes_S \cdots \otimes_S m_n \otimes_S m'_1 \otimes_S \cdots \otimes_S m'_t; \\ \eta(s) &: = (s, 0, 0, 0, \cdots). \end{aligned}$$

Notice that  $M \hookrightarrow \mathcal{T}(M)$ ,  $m \mapsto (0, m, 0, 0, \cdots)$ . In fact, we have an adjoint pair of functors  $(\mathcal{T}(-), \mathcal{U})$ , where  $\mathcal{U} : \mathbf{SAlg}_S \rightarrow \mathbb{S}_S$  is the forgetful functor. In other words,  $\mathcal{T}(M)$  satisfies the following universal property: given an  $S$ -linear map  $g : M \rightarrow A$ , where  $A$  is an  $S$ -semialgebra, there exists a morphism of  $S$ -semialgebras  $\tilde{g} : \mathcal{T}(M) \rightarrow A$  such that  $\tilde{g} \circ \iota = g$ . By this universal property, the  $S$ -linear maps

$$g : M \rightarrow \mathcal{T}(M) \otimes_S \mathcal{T}(M), \quad m \mapsto m \otimes_S 1 + 1 \otimes_S m \quad \text{and} \quad \mathfrak{z} : M \rightarrow S, \quad m \mapsto 0,$$

induce  $S$ -semialgebra morphisms

$$\Delta : \mathcal{T}(M) \rightarrow \mathcal{T}(M) \otimes_S \mathcal{T}(M) \quad \text{and} \quad \varepsilon : \mathcal{T}(M) \rightarrow S, \quad m \mapsto 0.$$

One can easily check that  $(\mathcal{T}(M), \mu, \eta, \Delta, \varepsilon)$  is an  $S$ -bisemialgebra.

**2.11.** Let  $C$  be an  $S$ -semicoalgebra. With a coideal of  $C$ , we mean an  $S$ -subsemimodule  $K \leq_S C$  such that  $K = \text{Ker}(f)$  for some *uniform* surjective morphism of  $S$ -semicoalgebras  $f : C \rightarrow C'$ . For characterizations of coideals of semicoalgebra over semirings, see [Abu-c, Proposition 2.16].

**Definition 2.12.** A *bi-ideal*  $I$  of an  $S$ -bisemialgebra  $B$  is an ideal of  $B^a$  which is also a coideal of  $B^c$ .

*Example 2.13.* Let  $B$  be an  $S$ -bisemialgebra such that  $\varepsilon_B$  is uniform. Notice that  $\varepsilon_B : B \rightarrow S$  is a surjective morphism of  $S$ -bisemialgebras, whence  $\text{Ker}(\varepsilon_B)$  is a bi-ideal.

**Lemma 2.14.** Let  $\gamma : B \rightarrow B'$  be a morphism of  $S$ -bisemialgebras.

1. If  $\gamma$  is surjective and uniform, then  $\text{Ker}(\gamma)$  is a bi-ideal.
2.  $\gamma(B)$  is an  $S$ -subbisemialgebra of  $B'$ .
3. For any bi-ideal  $I \subseteq \text{Ker}(\gamma)$ , there is a commutative diagram of  $S$ -bisemialgebras

$$\begin{array}{ccc} B & \xrightarrow{\gamma} & B' \\ & \searrow \pi_I & \nearrow f \\ & B/I & \end{array}$$

where  $\pi : B \rightarrow B/I$  is the canonical projection.

## Hopf Semimodules

In what follows, let  $(B, \mu, \eta, \Delta, \varepsilon)$  be an  $S$ -bisemialgebra.

**2.15.** If  $M, N \in \mathbb{S}_B$ , then  $M \otimes_S N$  has a *trivial* structure of a right  $B$ -semimodule  $(M \otimes_S N, - \otimes_S \rho^N)$  and another structure of a right  $B$ -semimodule  $(M \otimes_S^a N, \rho_{M \otimes_S^a N})$  where  $\rho_{M \otimes_S^a N}$  is the composition of the following maps

$$(M \otimes_S^a N) \otimes_S B \xrightarrow{- \otimes_S \Delta} (M \otimes_S^a N) \otimes_S (B \otimes_S B) \xrightarrow{- \otimes_S \tau_{(N, B)} \otimes_S -} (M \otimes_S B) \otimes_S (N \otimes_S B) \xrightarrow{\rho_{M \otimes_S^a N}} M \otimes_S^a N.$$

On the other hand, if  $M, N \in \mathbb{S}^B$ , then  $M \otimes_S N$  has a *trivial* structure of a right  $B$ -semicomodule  $(M \otimes_S N, - \otimes_S \rho^N)$  and another structure of a right  $B$ -semicomodule  $(M \otimes_S^c N, \rho^{M \otimes_S^c N})$  where  $\rho^{M \otimes_S^c N}$  is the composition of the following maps

$$M \otimes_S^c N \xrightarrow{\rho^{M \otimes_S^c N}} (M \otimes_S B) \otimes_S (N \otimes_S B) \xrightarrow{- \otimes_S \tau_{(B, N)} \otimes_S -} (M \otimes_S^c N) \otimes_S (B \otimes_S B) \xrightarrow{- \otimes_S \mu} (M \otimes_S^c N) \otimes_S B.$$

**2.16.** A *right-right Hopf semimodule* over  $B$  is a triple  $(M, \rho_M, \rho^M)$  such that  $(M, \rho_M)$  is a right  $B$ -semimodule,  $(M, \rho^M)$  is a right  $B$ -semicomodule and  $\rho_M : M \otimes_S^c B \rightarrow M$  is a morphism of right  $B$ -semicomodules, or equivalently  $\rho^M : M \rightarrow M \otimes_S^a B$  is a morphism of right  $B$ -semimodules, *i.e.*

$$\sum (mb)_{<0>} \otimes_S (mb)_{<1>} = \sum m_{<0>} b_1 \otimes_S m_{<1>} b_2 \text{ for all } m \in M, b \in B.$$

The category of right-right Hopf  $B$ -semimodules with arrows being the  $B$ -linear  $B$ -colinear maps is denoted by  $\mathbb{S}_B^B$ , *i.e.*  $\text{Hom}_B^B(M, N) = \text{Hom}_B(M, N) \cap \text{Hom}^B(M, N)$  for all  $M, N \in \mathbb{S}_B^B$ . Symmetrically, one can define the category  ${}^B\mathbb{S}_B$  of *right-left Hopf semimodules*, the category  ${}^B\mathbb{S}$  of *left-left Hopf  $B$ -semimodules* and the category  ${}_B\mathbb{S}^B$  of *left-right Hopf  $B$ -semimodules*.

*Remarks 2.17.* Let  $B$  be an  $S$ -bisemialgebra.

1.  $B \otimes_S^a B$  and  $B \otimes_S^c B$  are subgenerators in  $\mathbb{S}_B^B$  (cf. [BW2003, 14.5]).
2. Let  $M, N \in \mathbb{S}_B^B$ . Since  $\mathbb{S}_S$  has equalizers, we have

$$\text{Equal}(\varphi, \psi) = \text{Hom}_B^B(M, N) = \text{Equal}(\varkappa, \omega)$$

where

$$\varphi(f) = \rho^N \circ f, \quad \psi(f) = (f \otimes_S B) \circ \rho^M, \quad \varkappa(g) = \rho_N \circ (g \otimes_S B) \text{ and } \omega(g) = g \circ \rho_M.$$

$$\text{Hom}_B^B(M, N) \longrightarrow \text{Hom}_B(M, N) \xrightleftharpoons[\psi]{\varphi} \text{Hom}_B(M, N \otimes_S^a B)$$

$$\text{Hom}_B^B(M, N) \longrightarrow \text{Hom}_B(M, N) \xrightleftharpoons[\omega]{\varkappa} \text{Hom}_B(M, N \otimes_S^c B)$$

**Notation.** For  $N \in \mathbb{S}_B$  and  $L \in \mathbb{S}^B$ , we have the following morphisms in  $\mathbb{S}_B^B$ :

$$\begin{aligned} \gamma_N &: N \otimes_S B \longrightarrow N \otimes_S^a B, \quad n \otimes_S b \mapsto (n \otimes_S 1_B) \Delta(b) = \sum n b_1 \otimes_S b_2; \\ \gamma^L &: L \otimes_S^c B \longrightarrow L \otimes_S B, \quad l \otimes_S^c b \mapsto \rho^L(l)(1_B \otimes_S b) = \sum l_{<0>} \otimes_S l_{<1>} b. \end{aligned}$$

In particular,  $\gamma_B : B \otimes_S B \rightarrow B \otimes_S^a B$  and  $\gamma^B : B \otimes_S^c B \rightarrow B \otimes_S B$  are morphisms in  $\mathbb{S}_B^B$ .

**Definition 2.18.** Let  $\mathfrak{A}$  be a category with finite limits and finite colimits. A functor  $F : \mathfrak{A} \rightarrow \mathfrak{B}$  is said to be *left-exact* (*right-exact*) iff  $F$  preserves finite limits (finite colimits). Moreover,  $F$  is called *exact* iff  $F$  is left-exact and right-exact.

The following technical result will be needed in the sequel; the proof is straightforward.

**Lemma 2.19.** *Let  $(B, \mu, \eta, \Delta, \varepsilon)$  be an  $S$ -bisemialgebra.*

1.  $(\mathcal{F}, - \otimes_S^a B)$  is an adjoint pair of functors, where  $\mathcal{F} : \mathbb{S}_B^B \rightarrow \mathbb{S}_B$  is the forgetful functor. Consequently,  $- \otimes_S^a B : \mathbb{S}_B \rightarrow \mathbb{S}_B^B$  is left exact and preserves all limits (e.g. equalizers, kernels and direct products).
2.  $(- \otimes_S^c B, \mathcal{G})$  is an adjoint pair of functors, where  $\mathcal{G} : \mathbb{S}_B^B \rightarrow \mathbb{S}^B$  is the forgetful functor. Consequently,  $- \otimes_S^c B : \mathbb{S}^B \rightarrow \mathbb{S}_B^B$  is right exact and preserves all colimits (e.g. coequalizers, cokernels and direct coproducts).

**Proof.** It is a well-known fact that a left (right) adjoint functor is right (left) exact and preserves all colimits (limits).

1. For every  $M \in \mathbb{S}_B^B$  and  $N_B$ , we have a natural isomorphism of  $S$ -semimodules

$$\mathrm{Hom}_B^B(M, N \otimes_S^a B) \longrightarrow \mathrm{Hom}_B(\mathcal{F}(M), N), \quad f \mapsto (\vartheta_N^r \circ (N \otimes_S \varepsilon)) \circ f \quad (5)$$

with inverse  $h \mapsto [h \otimes_S B \circ \rho^M]$ .

2. For every  $M \in \mathbb{S}_B^B$  and  $N^B$ , we have a natural isomorphism of  $S$ -semimodules

$$\mathrm{Hom}_B^B(N \otimes_S^c B, M) \longrightarrow \mathrm{Hom}^B(N, \mathcal{G}(M)), \quad g \mapsto g(- \otimes_S 1_B) \quad (6)$$

with inverse  $h \mapsto [\rho_M \circ (h \otimes_S B)]$ .

## Integrals

As before, we let  $(B, \mu, \eta, \Delta, \varepsilon)$  be an  $S$ -bisemialgebra.

**2.20.** A *left (total) integral on  $B$*  is a left  $B$ -colinear morphism  $t \in B^* := \mathrm{Hom}_S(B, S)$  (with  $t(1_B) = 1_S$ ), equivalently  $\sum b_1 t(b_2) = 1_B t(b)$  for every  $b \in B$  (and  $t(1_B) = 1_S$ ). The *right (total) integrals* are defined symmetrically. For a justification of the terminology, we refer to [Mon1993] (see also [DNR2001, p. 181]). With  $\int^l \leq B^*$  ( $\int^{l,1} \leq B^*$ ) we denote the  $S$ -subsemimodule of (total) left integrals on  $B$ ; symmetrically, we denote with  $\int^r \leq B^*$  ( $\int^{r,1} \leq B^*$ ) the  $S$ -subsemimodule of (total) right integrals on  $B$ .

**Definition 2.21.** Consider  $B^*$  as a  $(B, B)$ -bisemimodule in the canonical way. If  ${}_S B$  is an  $\alpha$ -semimodule, then we call  $\mathrm{Rat}^B({}_B B^*)$  ( ${}^B \mathrm{Rat}(B_B^*)$ ) the *left (right) trace ideal* of  $B^*$ .

**Lemma 2.22.** *Let  $t \in B^* := \mathrm{Hom}_S(B, S)$ .*

1. *Assume that  $B$  is  $S$ -cogenerated in  $\mathbb{S}_S$ .*

(a)  *$t \in \int^l$  if and only if  $f * t = f(1_B)t$  for every  $f \in B^*$ .*

(b)  *$\int^l$  is an ideal of  $B^*$ .*

2. Let  ${}_S B$  be an  $\alpha$ -semimodule and  $B^{*rat} := \text{Rat}^B(B^* B^*)$ . Then  $t \in \int^l$  if and only if  $\rho^{B^{*rat}}(t) = t \otimes_S 1_B$ .

*Example 2.23.* Consider the polynomial  $S$ -bisemialgebra  $S[x]$  with  $\Delta(x) = x \otimes_S x$  and  $\varepsilon(x) = 1_S$ . It is clear that

$$t : S[x] \longrightarrow S, \quad p(x) \mapsto \delta_{1,p(x)}$$

is a *total* left (right) integral on  $S[x]$ : let  $f \in S[x]^*$ . For every  $n \in \mathbb{N}_0$ , we have

$$(f * t)(x^n) = f(x^n)t(x^n) = \delta_{1,x^n}f(x^n) = \delta_{0,n}f(x^n) = f(1)\delta_{0,n} = f(1)\delta_{1,x^n}.$$

Since  $S[x] \simeq S^{(\mathbb{N}_0)} \hookrightarrow S^{\mathbb{N}_0}$ , it is  $S$ -cogenerated as an  $S$ -semimodule, we conclude that  $t$  is a left integral on  $(S[x], \Delta, \varepsilon)$ . Similarly,  $t$  is a right integral on  $(S[x], \Delta, \varepsilon)$ .

*Example 2.24.* Consider the bisemialgebra  $B = \mathbb{B}[x]$ , where  $B$  is the *Boolean semifield*, with

$$\begin{aligned} \Delta(x) &= x \otimes_{\mathbb{B}} 1 + 1 \otimes_{\mathbb{B}} x, \quad \Delta(1) = 1 \otimes_S 1; \\ \varepsilon(x) &= 0, \quad \varepsilon(1) = 1. \end{aligned}$$

Let  $t \in B^*$  be a left integral on  $B$ . We have  $1_B t(x) = xt(1) + 1_B t(x)$ , whence  $t(1) = 0$ . We prove *by induction* that  $t(x^n) = 0$  for  $n \geq 1$ . Since  $\Delta(x^2) = x^2 \otimes_{\mathbb{B}} 1_B + x \otimes_{\mathbb{B}} x + 1_B \otimes_{\mathbb{B}} x^2$ , it follows that  $1_B t(x^2) = xt(x) + 1_B t(x)$ , whence  $t(x) = 0$ . Consider  $n \geq 2$  and assume that  $t(x^{n-i}) = 0$  for all  $i = 1, \dots, n$ . We have  $\Delta(x^{n+1}) = x^{n+1} \otimes_{\mathbb{B}} 1 + x^n \otimes_{\mathbb{B}} x + \dots + x \otimes_{\mathbb{B}} x^n + 1 \otimes_{\mathbb{B}} x^{n+1}$  and so  $1_B t(x^{n+1}) = xt(x^n) + 1_B t(x^{n+1})$ , whence  $t(x^n) = 0$ . Consequently,  $\int^l B = 0$ . Similarly, we can prove that  $\int^r B = 0$ .

**2.25.** Let  $M$  be a left  $B$ -semimodule. The set of *invariants* of  $M$  is

$${}^B M := \{m \in M \mid bm = \varepsilon(b)m \text{ for every } b \in B\}.$$

Moreover, we have an isomorphism of  $S$ -semimodules

$$\text{Hom}_{B-}(S, M) \longrightarrow {}^B M, \quad f \longmapsto f(1_S).$$

**Definition 2.26.** A *left integral* **in**  $B$  is an invariant of  ${}_B B$ , i.e. an element of

$${}^B B := \{\varpi \in B \mid b\varpi = \varepsilon(b)\varpi \text{ for every } \varpi \in B\}.$$

We say that a left integral  $\varpi$  in  $H$  is *normalized* iff  $\varepsilon(\varpi) = 1_S$ . Symmetrically, one can define (*normalized*) *right integrals* in  $B$ . With  $\int_l B$  ( $\int_{l,1} B$ ) we denote the set of (normalized) left integral in  $B$  and with  $\int_r B$  ( $\int_{r,1} B$ ) we denote the set of (normalized) right integrals in  $B$ .

*Example 2.27.* Let  $G = \{g_1, \dots, g_n\}$  be a finite group and consider the  $S$ -bisemialgebra  $B = S[G]$  with  $\Delta(g_i) = g_i \otimes_S g_i$  and  $\varepsilon(g_i) = 1_S$  for  $i = 1, \dots, n$ . Then  $\varpi := g_1 + \dots + g_n$  is a left (right) integral in  $B$ : indeed, for every  $g \in G$ , we have

$$g\varpi = g\left(\sum_{i=1}^n g_i\right) = \sum_{i=1}^n (gg_i) = \sum_{j=1}^n g_j = \varepsilon(g)\varpi.$$

**2.28.** Let  $M$  be a right  $B$ -semicomodule. The set of *coinvariants* of  $M$  is

$$M^{\text{co}B} := \text{Eq}(\rho^M, g) = \{m \in M \mid \rho^M(m) = m \otimes_S 1_B\}, \quad \text{where } g(m) := m \otimes_S 1_B$$

$$M^{\text{co}B} \longrightarrow M \xrightleftharpoons[g]{\rho^M} M \otimes_S B$$

**Lemma 2.29.** 1. For all  $L \in \mathbb{S}_S$  and  $M \in \mathbb{S}^B$ , we have an isomorphism

$$\mathrm{Hom}^B(L, M) \simeq \mathrm{Hom}_S(L, M^{\mathrm{co}B}). \quad (7)$$

2. We have an isomorphism of  $S$ -semimodules

$$\mathrm{Hom}^B(S, M) \longrightarrow M^{\mathrm{co}B}, \quad f \longmapsto f(1_S). \quad (8)$$

3.  $B^{\mathrm{co}B} = S1_B$ .

**Proposition 2.30.** Let  ${}_S B$  be an  $\alpha$ -semimodule and consider the trace ideal  $B^{*\mathrm{rat}} := \mathrm{Rat}^B(B^* B^*)$ .

1. For every  $M \in \mathbb{S}^B$ , we have  $M^{\mathrm{co}B} = {}^{B^*}M$ .
2.  $(B^{*\mathrm{rat}})^{\mathrm{co}B} = {}^{B^*}(B^{*\mathrm{rat}})$ .
3. If  ${}_S B$  is finitely generated and projective, then  $(B^*)^{\mathrm{co}B} = {}^{B^*}B^*$ .

**Proof.** (1) The proof is similar to that of [BW2003, 14.13 (1)].

(2) Set  $M = B^{*\mathrm{rat}}$  in (1) and notice that  $B^{*\mathrm{rat}} \in \mathbb{S}^B$  by (4).

(3) Since  ${}_S B$  is finitely generated and projective,  $B \simeq B^{**}$  and  $B^*$  is finitely generated and projective. It follows that we have a canonical isomorphism of  $S$ -semimodules  $B^* \otimes_S B^{**} \xrightarrow{\beta_{B^*}} \mathrm{Hom}_S(B^*, B^*)$  [KN2011, Proposition 3.7]. Notice that  $\alpha_{B^*}^B$  is in fact the composition of the following isomorphisms  $B^* \otimes_S B \simeq B^* \otimes_S B^{**} \simeq \mathrm{Hom}_S(B^*, B^*)$ , whence

$$B^{*\mathrm{rat}} := \mathrm{Rat}^B(B^* B^*) = \tilde{\rho}_{B^*}^{-1}(\alpha_{B^*}^B(B^* \otimes_S B)) = \tilde{\rho}_{B^*}^{-1}(\mathrm{Hom}_S(B^*, B^*)) = B^*$$

and the result follows from (2). ■

**Proposition 2.31.** 1.  $(- \otimes_S B, (-)^{\mathrm{co}B})$  is an adjoint pair of functors.

2.  $(- \otimes_S B, \mathrm{Hom}_B^B(B, -))$  is an adjoint pair of functors.

3. We have a natural isomorphism of functors

$$\mathrm{Hom}_B^B(B, -) \simeq (-)^{\mathrm{co}B}. \quad (9)$$

4.  $\mathrm{Hom}_B^B(B, -)$  is left exact and preserves all limits (e.g. equalizers, kernels and direct products).

5. We have an isomorphism of semirings

$$\mathrm{End}_B^B(B) \simeq B^{\mathrm{co}B} = B1_S. \quad (10)$$

**Proof.** It is clear that  $(-)^{\mathrm{co}B} : \mathbb{S}_B^B \longrightarrow \mathbb{S}_S$  and  $\mathrm{Hom}_B^B(B, -) : \mathbb{S}_B^B \longrightarrow \mathbb{S}_S$  are functors.

1. For every  $M \in \mathbb{S}_S$  and  $N \in \mathbb{S}_B^B$ , we have a natural isomorphism

$$\mathrm{Hom}_B^B(M \otimes_S B, N) \simeq \mathrm{Hom}_S(M, N^{\mathrm{co}B}), \quad f \mapsto [m \mapsto f(m \otimes_S 1_B)] \quad (11)$$

with inverse  $g \mapsto [m \otimes_S b \mapsto g(m)b]$ .

2. For every  $M \in \mathbb{S}_S$  and  $N \in \mathbb{S}_B^B$ , we have a natural isomorphism

$$\mathrm{Hom}_B^B(M \otimes_S B, N) \simeq \mathrm{Hom}_S(M, \mathrm{Hom}_B^B(B, N)), \quad f \mapsto [- \mapsto f(- \otimes_S 1_B)] \quad (12)$$

with inverse  $g \mapsto [m \otimes_S b \mapsto g(m)(b)]$ .

3. This follows from (1) and (2) by the uniqueness of the right adjoint of the functor  $- \otimes_S B : \mathbb{S}_S \longrightarrow \mathbb{S}_B^B$ . In fact, substituting  $M = S$  in (11) yields a natural isomorphism for every  $N \in \mathbb{S}_B^B$ :

$$\mathrm{Hom}_B^B(B, N) \simeq N^{\mathrm{co}B}, \quad f \mapsto f(1_B) \quad (13)$$

with inverse  $n \mapsto [b \mapsto nb]$ .

4. This follows directly from the fact that  $\mathrm{Hom}_B^B(B, -)$  has a left adjoint by (2).

5. Set  $N := B$  in (2). It is clear that the isomorphism obtained is in fact a morphism of semirings. ■

We present now the main reconstruction result in this section:

**Theorem 2.32.** 1. We have an isomorphism of categories  $\mathbf{SBialg}_S \simeq \mathbf{Bimonoid}(\mathbb{S}_S)$ .

2. Let  $B$  be an  $S$ -semimodule. There is a bijective correspondence between the structures of  $S$ -bisemialgebras on  $B$ , the bimonad structures on  $B \otimes_S - : \mathbb{S}_S \longrightarrow \mathbb{S}_S$  and the bimonad structures on  $- \otimes_S B : \mathbb{S}_S \longrightarrow \mathbb{S}_S$ .

3. We have isomorphisms of categories

$$\begin{aligned} \mathbb{S}_B^B &\simeq (\mathbb{S}_B)^{-\otimes_S B} \simeq ((\mathbb{S}_S)_{-\otimes_S B})^{-\otimes_S B}; \\ &\simeq (\mathbb{S}^B)_{-\otimes_S B} \simeq ((\mathbb{S}_S)^{-\otimes_S B})_{-\otimes_S B}. \end{aligned}$$

**Proof.** (1) and (3) follow directly from the definitions. The proof of (2) is along the lines of that of [Ver, Theorem 3.9] taking into consideration that  $\mathbb{S}_S$  is cocomplete, that  $S$  is a regular generator in  $\mathbb{S}_S$  [Gol1999] and the fact that  $- \otimes_S X \simeq X \otimes_S - : \mathbb{S}_S \longrightarrow \mathbb{S}_S$  preserves colimits for every  $S$ -semimodule  $X$ . ■

## An Application

**2.33.** ([Wor2012]) A *right  $S$ -linear automaton* is a datum  $\mathbf{A} = (M, A, \mathbf{s}, \boldsymbol{\rho}, \Omega)$ , where  $A$  is an  $S$ -semialgebra,  $M$  is a right  $A$ -semimodule,  $\mathbf{s} \in M$  (called a *starting vector*) and  $\Omega \in \mathrm{Hom}_S(M, S)$  (called an *observation function*). The *language accepted by a right  $S$ -linear automaton  $A$*  is the  $S$ -linear map  $\boldsymbol{\rho} : A \longrightarrow S$ ,  $a \mapsto \Omega(\mathbf{s}a)$ .

**2.34.** Let  $B$  be an  $S$ -bisemialgebra. If  $\mathbf{A} = (M, B, \mathbf{s}, \rho, \Omega)$  and  $\mathbf{A}' = (M', B, \mathbf{s}', \rho', \Omega')$  are two left  $S$ -linear automata, then  $\mathbf{A} \otimes_S \mathbf{A}' := (M \otimes_S^b N, B, \mathbf{s} \otimes_S \mathbf{s}', \rho_{M \otimes_S^b N}, \Omega \otimes_S \Omega')$  is a right  $S$ -linear automaton and the language accepted by  $A \otimes_S A'$  is  $\boldsymbol{\rho}_{A \otimes_S A'} := \boldsymbol{\rho} * \boldsymbol{\rho}'$ .



### 3 Doi-Koppinen Semimodules

The class of Doi-Koppinen modules over rings was introduced independently by Y. Doi [Doi1992] and M. Koppinen [Kop1995]. In this section, we extend these notions and some results on them to Doi-Koppinen semimodules over semirings. Throughout this section,  $(B, \mu, \eta, \Delta, \varepsilon)$  is an  $S$ -bisemialgebra.

**Definition 3.1.** 1. A *right  $B$ -semimodule semialgebra* is an  $S$ -semialgebra  $(A, \mu_A, \eta_A)$  with a right  $B$ -semimodule structure such that  $\mu_A$  and  $\eta_A$  are  $B$ -linear, i.e.

$$(a\tilde{a})b = \sum (ab_1)(\tilde{a}b_2) \text{ and } 1_A b = \varepsilon_B(b)1_A \text{ for all } a, \tilde{a} \in A \text{ and } b \in B. \quad (14)$$

Symmetrically, one defines a *left  $B$ -semimodule algebra*.

2. A *right  $B$ -semimodule semicoalgebra* is an  $S$ -semicoalgebra  $(C, \Delta_C, \varepsilon_C)$  with a right  $B$ -semimodule structure such that  $\Delta_C$  and  $\varepsilon_C$  are  $B$ -linear, i.e.

$$\sum (cb)_1 \otimes_S (cb)_2 = \sum c_1 b_1 \otimes c_2 b_2 \text{ and } \varepsilon_C(cb) = \varepsilon_C(c)\varepsilon_B(b) \text{ for all } c \in C \text{ and } b \in B. \quad (15)$$

Symmetrically, one defines a *left  $B$ -semimodule semicoalgebra*.

3. A *right  $B$ -semicomodule semialgebra* is an  $S$ -semialgebra  $(A, \mu_A, \eta_A)$  with a right  $B$ -semicomodule structure such that  $\mu_A$  and  $\eta_A$  are  $B$ -colinear, i.e.

$$\sum (ab)_{<0>} \otimes_S (ab)_{<1>} = \sum a_{<0>} b_{<0>} \otimes_S a_{<1>} b_{<1>} \text{ and } \sum 1_{<0>} \otimes_S 1_{<1>} = 1_A \otimes 1_B. \quad (16)$$

Symmetrically, one defines a *left  $B$ -semicomodule semialgebra*.

4. A *right  $B$ -semicomodule semicoalgebra* is an  $S$ -semicoalgebra  $(C, \Delta_C, \varepsilon_C)$  with a right  $B$ -semicomodule structure such that  $\Delta_C$  and  $\varepsilon_C$  are  $B$ -colinear, i.e.

$$\sum c_{<0>1} \otimes c_{<0>2} \otimes c_{<1>} = \sum c_{1<0>} \otimes c_{2<0>} \otimes c_{1<1>} c_{2<1>} \text{ and } \sum \varepsilon_C(c_{<0>}) c_{<1>} = \varepsilon_C(c)1_B. \quad (17)$$

Symmetrically, one defines a *left  $B$ -semicomodule semicoalgebra*.

**3.2.** A *right-right Doi-Koppinen structure* over  $S$  is a triple  $(B, A, C)$  consisting of an  $S$ -bisemialgebra  $B$ , a right  $B$ -semicomodule semialgebra  $A$  and a right  $B$ -semimodule semicoalgebra  $C$ . A *right-right Doi-Koppinen semimodule* for  $(B, A, C)$  is a right  $A$ -semimodule  $M$ , which is also a right  $C$ -semicomodule such that

$$\sum (ma)_{<0>} \otimes_S (ma)_{<1>} = \sum m_{<0>} a_{<0>} \otimes_S m_{<1>} a_{<1>} \text{ for all } m \in M \text{ and } a \in A.$$

With  $\mathbb{S}(B)_A^C$  we denote the category of right-right Doi-Koppinen semimodules and  $A$ -linear  $C$ -colinear morphisms.

The following result is easy to prove.

**Lemma 3.3.** *Let  $(B, A, C)$  be a right-right Doi-Koppinen structure over  $S$ .*

1.  $\#^{op}(C, A) := \text{Hom}_S(C, A)$  is an  $A$ -semiring with  $(A, A)$ -bisemimodule structure

$$(af)(c) := \sum a_{<0>} f(ca_{<1>}) \text{ and } (fa)(c) := f(c)a,$$

*multiplication*

$$(f \cdot g)(c) = \sum f(c_2)_{<0>} g(c_1 f(c_2)_{<1>}) \quad (18)$$

*and unity  $\eta_A \circ \varepsilon_C$ .*

2.  $\mathcal{C} := A \otimes_S C$  is an  $A$ -semicoring and  $\#^{op}(C, A) \simeq {}^* \mathcal{C}$  as  $A$ -semirings.

*Examples 3.4.* 1.  $C = B$  is a right  $B$ -semimodule semicoalgebra with structure map  $\mu_B$  and so  $(B, A, B)$  is a right-right Doi-Koppinen structure for every right  $B$ -semicomodule semialgebra  $A$ . In this case,  $\mathbb{S}(B)_A^B = \mathbb{S}_A^B$ , the category of *relative Hopf semimodules* (cf. [Doi1983]).

2.  $A = B$  is a right  $B$ -semicomodule semialgebra with structure map  $\Delta_B$  and so  $(B, B, C)$  is a right-right Doi-Koppinen structure for every right  $B$ -semimodule semicoalgebra  $C$ . In this case,  $\mathbb{S}(B)_B^C = \mathbb{S}_{[C, B]}$ , the category of *Doi's  $[C, B]$ -semimodules* (cf. [Doi1983]).

3. Setting  $A = B = C$ , we notice that  $(B, B, B)$  is a right-right Doi-Koppinen structure and that  $\mathbb{S}(B)_B^B = \mathbb{S}_B^B$ , the category of *Hopf semimodules* (cf. [Swe1969, 4.1]).

The following result is easy to prove.

**Lemma 3.5.** *Let  $(B, A, C)$  be a right-right Doi-Koppinen structure over  $S$  and consider the corresponding  $A$ -semicoring  $\mathcal{C} := A \otimes_S C$ .*

1.  $A \#^{op} C^* := A \otimes_S C^*$  is an  $A$ -semiring with  $(A, A)$ -bisemimodule structure

$$\tilde{a}(a \# f) := \sum \tilde{a}_{<0>} a \# \tilde{a}_{<1>} f \text{ and } (a \# f) \tilde{a} := a \tilde{a} \# f, \quad (19)$$

multiplication

$$(a \# f) \cdot (b \# g) := \sum a_{<0>} b \# (a_{<1>} g) * f \quad (20)$$

and unity  $1_A \# \varepsilon_C$ . Moreover,

$$\eta : A \longrightarrow A \#^{op} C^*, a \mapsto a \# \varepsilon_C$$

is a morphism of  $A$ -semirings.

2.  $P := (A \#^{op} C^*, \mathcal{C})$  is a measuring left  $A$ -pairing (in the sense of [Abu-c]).

**Proof.** 1. Clear.

2. It is clear that

$$\kappa_P : A \#^{op} C^* \longrightarrow {}^* \mathcal{C}, a \# f \mapsto [\tilde{a} \otimes_S c \mapsto \tilde{a} a f(c)]$$

is a morphism of  $A$ -semirings. ■

**Theorem 3.6.** *Let  $(B, A, C)$  be a right-right Doi-Koppinen structure and consider the corresponding  $A$ -semicoring  $\mathcal{C} := A \otimes_S C$ . We have an isomorphism of categories*

$$\mathbb{S}(B)_A^C \simeq \mathbb{S}^{\mathcal{C}}.$$

**Proof.** It can be shown that  $(A, C, \psi)$  is a *right-right entwining structure* in the symmetric monoidal category  $(\mathbb{S}_S, \otimes_S, S; \tau)$ , where

$$\psi : C \otimes_A A \longrightarrow A \otimes_A C, c \otimes_A a \mapsto \sum a_{<0>} \otimes_A c a_{<1>}. \quad (21)$$

By arguments similar to those in [Brz1999] (see also [BW2003, Ch. 5]), one can show that  $\mathbb{S}(B)_A^C \simeq \mathbb{S}_A^C(\psi) \simeq \mathbb{S}^{\mathcal{C}}$ , where  $\mathbb{S}_A^C(\psi)$  is the associated category of *right-right entwined semimodules*. ■

**Proposition 3.7.** *Let  $(B, A, C)$  be a right-right Doi-Koppinen structure and the associated category  $\mathbb{S}(B)_A^C$  of right-right Doi-Koppinen semimodules.*

1.  $\mathbb{S}(B)_A^C$  is comonadic, locally presentable and a covariety (in the sense of [AP2003]).
2. The forgetful functor  $\mathcal{F} : \mathbb{S}(B)_A^C \longrightarrow \mathbb{S}_A$  creates all colimits and isomorphisms.
3.  $\mathbb{S}(B)_A^C$  is cocomplete, i.e.  $\mathbb{S}(B)_A^C$  has all (small) colimits, e.g. coequalizers, cokernels, pushouts, directed colimits and direct sums. Moreover, the colimits are formed in  $\mathbb{S}_A$ .
4.  $\mathbb{S}(B)_A^C$  is complete, i.e.  $\mathbb{S}(B)_A^C$  has all (small) limits, e.g. equalizers, kernels, pullbacks, inverse limits and direct products. Moreover, the forgetful functor  $\mathcal{F}$  creates all limits preserved by  $- \otimes_A (A \otimes_S C) : \mathbb{S}_A \longrightarrow \mathbb{S}_A$ .

**Proof.** The result is an immediate consequence of [Abu-c, Proposition 2.22] taken into consideration that  $\mathcal{C} := A \otimes_S C$  is an  $A$ -semicoring and that  $\mathbb{S}(B)_A^C \simeq \mathbb{S}^{\mathcal{C}}$ . ■

*Remark 3.8.* Let  $(B, A, C)$  be a right-right Doi-Koppinen structure over  $S$ . While it is guaranteed that the category  $\mathbb{S}(B)_A^C$  has kernels, these are not necessarily formed in  $\mathbb{S}_A$ . Indeed, if  ${}_S C$  is flat, then  ${}_A \mathcal{C}$  is flat and it follows by [Abu-c, Proposition 2.26] that all equalizers in  $\mathbb{S}(B)_A^C \simeq \mathbb{S}^{\mathcal{C}}$  are formed in  $\mathbb{S}_A$ .

**Theorem 3.9.** *Let  $(B, A, C)$  be a right-right Doi-Koppinen structure and consider the corresponding  $A$ -semicoring  $\mathcal{C} := A \otimes_S C$ . If  $P := (A \#^{\text{op}} C^*, \mathcal{C})$  satisfies the  $\alpha$ -condition, then we have an isomorphisms of categories*

$$\mathbb{S}(B)_A^C \simeq \text{Rat}^{\mathcal{C}}(\mathbb{S}_{A \#^{\text{op}} C^*}) = \sigma[\mathcal{C}_{A \#^{\text{op}} C^*}].$$

**Proof.** The isomorphism  $\mathbb{S}^{\mathcal{C}} \simeq \text{Rat}^{\mathcal{C}}(\mathbb{S}_{A \#^{\text{op}} C^*})$  follows by [Abu-c, Theorem 3.16]. A similar argument to that of [Abu-c, Theorem 3.22] shows that  $\mathbb{S}^{\mathcal{C}} \simeq \text{Rat}^{\mathcal{C}}(\mathbb{S}_{B \#^{\text{op}} B^*}) = \sigma[\mathcal{C}_{B \#^{\text{op}} B^*}]$ . ■

We provide now an example in which the assumption of Theorem 3.6 (2) holds:

*Example 3.10.* Let  $(B, A, C)$  be a right-right Doi-Koppinen structure with  ${}_S C$  an  $\alpha$ -semimodule. Consider the left measuring  $A$ -pairing  $P := (A \#^{\text{op}} C^*, \mathcal{C})$  and let  $\phi : C^* \longrightarrow A \otimes_S C^*$ ,  $f \mapsto 1_A \otimes_S f$ . For every right  $A$ -semimodule  $M$ , we have the following commutative diagram

$$\begin{array}{ccc} M \otimes_A (A \otimes_S C) & \xrightarrow{\alpha_M^P} & \text{Hom}_{-A}(A \otimes_S C^*, M) \\ \parallel & & \downarrow (\phi, M) \\ M \otimes_S C & \xrightarrow{\alpha_M^{\mathcal{C}}} & \text{Hom}_S(C^*, M) \end{array}$$

Since  $\alpha_M^{\mathcal{C}}$  is injective, it follows that  $\alpha_M^P$  is injective. We claim that  $\alpha_M^P$  is uniform for every  $M_A$ . Let  $M$  be a right  $A$ -semimodule and let  $h \in \overline{\alpha_M^P(M \otimes_A (A \otimes_S C))}$ , i.e.  $h + \alpha^P(\sum m_i \otimes_A (a_i \otimes_S c_i)) = \alpha^P(\sum m'_i \otimes_A (a'_i \otimes_S c'_i))$ . For every  $g \in C^*$  we have

$$\begin{aligned} h(1_A \otimes_S g) + \alpha_M^P(\sum m \otimes_A (a_i \otimes_S c_i))(1_A \otimes_S g) &= \alpha_M^P(\sum m_i \otimes_A (a_i \otimes_S c_i))(1_A \otimes_S g) \\ (\phi, M)(h)(g) + \alpha_M^{\mathcal{C}}(\sum m_i a_i \otimes_S c_i)(g) &= \alpha_M^{\mathcal{C}}(\sum m_i a_i \otimes_S c_i)(g), \end{aligned}$$

whence  $(\phi, M)(h) \in \overline{\alpha_M^C(M \otimes_S C)}$ . Since  $\alpha_M^C$  is uniform (by our assumption on  ${}_S C$ ), there exists  $\sum m'_j \otimes_S c'_j \in M \otimes_S C$   $(\varphi, M)(h)$  such that for every  $g \in C^*$  we have

$$h(1 \otimes g) = (\varphi, M)(h)(g) = \alpha^C(\sum m_j \otimes c_j)(g) = \sum m_j g(c_j).$$

Noting that  $h$  is right  $A$ -linear, it follows that for every  $\sum_l a'_l \otimes g'_l \in A \otimes_S C^*$  we have

$$\begin{aligned} h(\sum_l a'_l \otimes g'_l) &= h(\sum_l (1_A \otimes_S g'_l) a'_l) = \sum_l h(1_A \otimes_S g'_l) a'_l \\ &= \sum_k \sum_j m_j a'_k g'_k(c_j) = \alpha_M^P(\sum_j m_j \otimes_A (1_A \otimes_S c_j))(\sum_l a_l \otimes g_l), \end{aligned}$$

i.e.  $h \in \alpha_M^P(M \otimes_A (A \otimes_S C))$ . Consequently,  $\alpha_M^P$  is uniform.

**Corollary 3.11.** *Let  $B$  be an  $S$ -bisemialgebra.*

1. *If  ${}_S B$  is an  $\alpha$ -semimodule, then we have isomorphisms of categories*

$$\mathbb{S}_B^B \simeq \mathbb{S}^{B \otimes_S^a B} \simeq \text{Rat}^{B \otimes_S^a B}(\mathbb{S}_{B \#^{op} B^*}) = \sigma[B \otimes_S^a B_{B \#^{op} B^*}].$$

2. *If  ${}_S B$  is uniformly finitely presented and flat, then we have an isomorphism of categories*

$$\mathbb{S}_B^B \simeq \mathbb{S}_{B \#^{op} B^*}.$$

**Proof.** 1. This follows by Theorem 3.6 and Example 3.10.

2. Since  ${}_S B$  is uniformly finitely presented and flat, we have by Lemma 1.12 a canonical isomorphism  $B \otimes_S B^* \xrightarrow{v_{(B, B, S)}} \text{End}_S(B)$ . An argument similar to that of [BW2003, 3.11] shows that  $B^*$  is a left  $B$ -semicomodule through  $B^* \xrightarrow{\chi} \text{End}_S(B) \xrightarrow{v_{(B, B, S)}} B \otimes_S B^*$ , where  $\chi(g) = \vartheta_B^r \circ (B \otimes_S g) \circ \Delta_B$ . Simple computations show that the twisting map  $\tau_{(B, B)} : B^* \otimes_S^c B \rightarrow B \#^{op} B^*$  is right  $B \#^{op} B^*$ -linear, whence  $B \#^{op} B^* \in \mathbb{S}_B^B$  and so  $\mathbb{S}_B^B \simeq \mathbb{S}_{B \#^{op} B^*}$ . ■

## 4 Hopf Semialgebras

**4.1.** With a *Hopf  $S$ -semialgebra*, we mean a datum  $(H, \mu, \eta, \Delta, \varepsilon, \mathfrak{a})$ , where  $(H, \mu, \eta, \Delta, \varepsilon)$  is an  $S$ -bisemialgebra and  $\text{id}_H$  is a unit (invertible) in the endomorphism semiring  $(\text{End}_S(H), *)$ , i.e. there exists an  $S$ -linear map  $\mathfrak{a} : H \rightarrow H$  such that

$$\sum \mathfrak{a}(h_1) h_2 = \varepsilon(h) 1_H = \sum h_1 \mathfrak{a}(h_2) \text{ for all } h \in H.$$

A *morphism of Hopf  $S$ -semialgebras*  $f : H \rightarrow H'$  is a morphism of  $S$ -bisemialgebras which is compatible with the antipodes of  $H$  and  $H'$  in the sense that

$$\mathfrak{a}_{H'} \circ f = f \circ \mathfrak{a}_H. \quad (22)$$

The category of Hopf  $S$ -semialgebras is denoted by **HopfAlg** $_S$ .

*Remarks 4.2.* 1. If  $(H, \mu, \eta, \Delta, \varepsilon, \mathfrak{a})$  is a Hopf  $S$ -semialgebra, then  $\mathfrak{a} : H \rightarrow H$  is a bialgebra anti-morphism.

2. The category  $\mathbf{HopAlg}_S \hookrightarrow \mathbf{Bialg}_S$  is a full subcategory, *i.e.* if  $H$  and  $H'$  are Hopf  $S$ -semialgebras and  $f : H \rightarrow H'$  is an  $S$ -bialgebra morphism, then  $f$  is a morphism of Hopf  $S$ -semialgebras.
3. An  $S$ -bisemialgebra  $B$  is a Hopf  $S$ -semialgebra with invertible antipode  $\mathbf{a}$  if and only if  $B^{cop}$  is a Hopf  $S$ -semialgebra with invertible antipode  $\tilde{\mathbf{a}}$ . Moreover, in this  $\mathbf{a}^{-1} = \tilde{\mathbf{a}}$ . In particular, if  $H$  is a commutative (cocommutative) Hopf  $S$ -semialgebra, then  $\mathbf{a}_H^2 = \text{id}$  (cf. [Mon1993, Lemma 1.5.11, Corollary 1.5.12]).

**Definition 4.3.** Let  $(H, \mu, \eta, \Delta, \varepsilon, \mathbf{a})$  be a Hopf  $S$ -semialgebra. A bi-ideal  $I \leq_S H$  is said to be a *Hopf ideal* iff  $\mathbf{a}(I) \subseteq I$ .

**Proposition 4.4.** Let  $(H, \mu, \eta, \Delta, \varepsilon, \mathbf{a})$  be a Hopf  $S$ -semialgebra. For every Hopf ideal  $I \leq_S H$ , we have a Hopf algebra structure on  $H/I$ ; moreover, the canonical projection  $\pi_I : H \rightarrow H/I$  is a morphism of Hopf  $S$ -semialgebras.

*Example 4.5.* Let  $(G, \cdot, e)$  be a group and consider the  $S$ -bisemialgebra  $S[G]$  (Example 2.5). One can easily see that

$$\mathbf{a} : S[G] \rightarrow S[G], \quad g \mapsto g^{-1}$$

is an antipode for  $S[G]$ . So,  $S[G]$  is a Hopf  $S$ -semialgebra.

*Example 4.6.* Consider the  $S$ -semialgebra  $S[x, x^{-1}]$  with the usual multiplication and unity. Notice that  $G = \{x^z \mid z \in \mathbb{Z}\}$  is a group and that  $S[x, x^{-1}] \simeq S[G]$  as  $S$ -semialgebras. It follows that  $(S[x, x^{-1}], \mu, \eta, \Delta, \varepsilon, \mathbf{a})$  is a Hopf  $S$ -semialgebra, where the structure maps are defined by extending the following assignments linearly

$$\begin{array}{lll} \Delta & : & S[x, x^{-1}] \longrightarrow S[x, x^{-1}] \otimes_S S[x, x^{-1}], \quad x^z \mapsto x^z \otimes_S x^z; \\ \varepsilon & : & S[x, x^{-1}] \longrightarrow S, \quad x^z \mapsto 1_S; \\ \mathbf{a} & : & S[x, x^{-1}] \longrightarrow S[x, x^{-1}], \quad x^z \mapsto x^{-z}. \end{array}$$

*Example 4.7.* Let  $(S, \oplus, \cdot, \mathbf{0}, \mathbf{1})$  be an  $S$ -semiring which has no non-zero zerodivisors and let  $\mathbf{0} \neq \mathbf{2} = ab$  in  $S$ , *e.g.*  $S$  is an additively idempotent semiring (Remark 1.2). One can easily see that  $H := S[x]/(bx + x^2)$  is a Hopf  $S$ -semialgebra with

$$\begin{array}{ll} \Delta & : H \longrightarrow H \otimes_S H, \quad \bar{x} \mapsto \bar{x} \otimes_S 1_H \oplus 1_H \otimes_S \bar{x} \oplus a\bar{x} \otimes_S \bar{x}; \\ \varepsilon & : H \longrightarrow S, \quad x \mapsto \mathbf{0}; \\ \mathbf{a} & : H \longrightarrow H, \quad \bar{x} \mapsto \bar{x}. \end{array}$$

Notice that

$$\begin{aligned} \sum \mathbf{a}(\bar{x}_1)\bar{x}_2 &= \mathbf{a}(\bar{x})1_H \oplus \mathbf{a}(1_H)\bar{x} \oplus \mathbf{a}(a\bar{x})\bar{x} &= \bar{x} \oplus \bar{x} \oplus \overline{ax^2} \\ &= \mathbf{2}\bar{x} \oplus \overline{ax^2} &= (a \cdot b)\bar{x} \oplus \overline{ax^2} \\ &= a(\overline{b\bar{x} \oplus \bar{x}^2}) &= a(\overline{bx + x^2}) \\ &= a(0_H) &= 0_H \\ &= \mathbf{0}(1_H) &= \varepsilon(\bar{x})1_H. \end{aligned}$$

Similarly,  $\sum_1 \bar{x}_1 \mathbf{a}(\bar{x}_2) = \varepsilon(\bar{x})1_H$ . Consequently,  $S[x]/(bx + x^2)$  is a Hopf  $S$ -semialgebra. ■

## Quantum Monoids

**Definition 4.8.** A *quantum monoid* is a non-commutative non-cocommutative Hopf semialgebra.

*Example 4.9.* Consider the  $S$ -semialgebra with four *different* generators

$$H = S \langle 1, g, x, y \mid g^2 = 1, x^2 = xy = yx = y^2 = 0, xg = gy, yg = gx, x + y = 0 \rangle.$$

Notice that  $H$  is an  $S$ -bisemialgebra with comultiplication and counity obtained by extending the following assignments as  $S$ -semialgebra morphisms

$$\begin{aligned} \Delta(1) &= 1 \otimes_S 1, \Delta(g) = g \otimes_S g, \Delta(x) = x \otimes_S 1 + g \otimes_S x, \Delta(y) = y \otimes_S 1 + g \otimes_S y; \\ \varepsilon(1) &= 1 = \varepsilon(g), \varepsilon(x) = 0 = \varepsilon(y). \end{aligned}$$

Moreover,  $H$  is a Hopf  $S$ -semialgebra with antipode defined by extending the following assignments linearly

$$\mathfrak{a}(1) = 1, \mathfrak{a}(g) = g, \mathfrak{a}(x) = xg, \mathfrak{a}(y) = yg.$$

Clearly,  $H$  is *non-commutative* and *non-cocommutative*, i.e.  $H$  is a quantum monoid. Notice that  $H$  is in fact a semialgebraic version of Sweedler's Hopf Algebra (quantum group).

*Example 4.10.* Let  $n \geq 2$  and  $q \in S$  be such that  $q^n = 1_S$  and  $q^i \neq 1$  for any  $i \in \{1, \dots, n-1\}$ . Consider the  $S$ -semialgebra with four *different* generators

$$H = S \langle 1, g, x, y \mid g^n = 1, x^i y^{n-i} = 0 = y^{n-i} x^i, i = 0, \dots, n, xg = qgx, yg = qgy, x + y = 0 \rangle.$$

Notice that  $H$  is an  $S$ -bisemialgebra with comultiplication and counity obtained by extending the following assignments as  $S$ -semialgebra morphisms

$$\begin{aligned} \Delta(1) &= 1 \otimes_S 1, \Delta(g) = g \otimes_S g, \Delta(x) = x \otimes_S 1 + g \otimes_S x, \Delta(y) = y \otimes_S 1 + g \otimes_S y; \\ \varepsilon(1) &= 1 = \varepsilon(g), \varepsilon(x) = 0 = \varepsilon(y). \end{aligned}$$

Moreover,  $H$  is a Hopf  $S$ -semialgebra with antipode defined by extending the following assignments linearly

$$\mathfrak{a}(1) = 1, \mathfrak{a}(g) = g^{-1}, \mathfrak{a}(x) = g^{-1}y, \mathfrak{a}(y) = g^{-1}x.$$

Clearly,  $H$  is *non-commutative* and *non-cocommutative*, i.e.  $H$  is a quantum monoid. Notice that  $H$  is in fact a semialgebraic version of Taft's Hopf Algebra (quantum group).

*Example 4.11.* The  $S$ -semialgebra  $H = S[x, y, y^{-1}]/(xy + yx, x^2)$ , where  $x$  and  $y$  are *non-commuting* indeterminates, is a quantum monoid with

$$\begin{aligned} \Delta(1) &= 1 \otimes_S 1, \Delta(x) = x \otimes_S 1 + y^{-1} \otimes_S x, \Delta(y) = y \otimes_S y; \\ \varepsilon(1) &= 1, \varepsilon(x) = 0, \varepsilon(y) = 1; \\ \mathfrak{a}(1) &= 1, \mathfrak{a}(x) = xy, \mathfrak{a}(y) = y^{-1}, \mathfrak{a}(y^{-1}) = y. \end{aligned}$$

In fact,  $H$  is a semialgebraic version of Pareigis' Hopf algebra (quantum group) [HGK2010, Example 3.4.22].

## Fundamental Theorem

In what follows we give sufficient and necessary conditions for a given  $S$ -bisemialgebra to have an antipode (*i.e.* to be a Hopf  $S$ -semialgebra).

**Proposition 4.12.** *Let  $B$  be an  $S$ -bisemialgebra. The following are equivalent:*

1.  $B$  is a Hopf  $S$ -semialgebra;
2.  $B \otimes_S B \xrightarrow{\gamma_B} B \otimes_S^a B$  is an isomorphism in  $\mathbb{S}_B^B$ , where  $\gamma_B(a \otimes_S b) = \sum ab_1 \otimes_S^a b_2$ ;
3.  $B \otimes_S^c B \xrightarrow{\gamma_B} B \otimes_S B$  is an isomorphism in  $\mathbb{S}_B^B$ , where  $\gamma^B(a \otimes_S^c b) = \sum a_1 \otimes_S a_2 b$ .

**Proof.** The proof is based on direct standard computations (*e.g.* [Ion1998]). Since, our semimodules do not allow subtraction (in general), we notice here that the proof of [BW2003, 15.2 (3)] does *not* work in our case. We prove (1)  $\iff$  (2); notice that (1)  $\iff$  (3) can be proved symmetrically.

(1)  $\Rightarrow$  (2) If  $\mathfrak{a}$  is an antipode for  $B$ , then it is clear that  $\gamma_B$  is an isomorphism with inverse

$$\omega_B : B \otimes_S^a B \longrightarrow B \otimes_S B, \quad a \otimes_S^a b \mapsto \sum a\mathfrak{a}(b_1) \otimes_S b_2.$$

(2)  $\Rightarrow$  (1) Assume that  $\gamma_B$  is invertible with inverse  $\omega_B : B \otimes_S B \longrightarrow B \otimes_S^a B$  in  $\mathbb{S}_B^B$ ; in particular,  $\omega_B$  is right  $B$ -colinear, *i.e.*

$$(\omega_B \otimes_S B) \circ (B \otimes_S \Delta) = (B \otimes_S \Delta) \circ \omega_B \tag{23}$$

Moreover, since  $\gamma_B$  is left  $B$ -linear, its inverse  $\omega_B$  is indeed left  $B$ -linear. Setting

$$\xi_B^r : B \otimes_S B \xrightarrow{B \otimes_S \varepsilon} B \otimes_S S \xrightarrow{\vartheta_B^r} B \text{ and } \omega_B(1 \otimes_S b) = \sum b^{(0)} \otimes_S b^{(1)},$$

we claim that

$$\mathfrak{a} : B \longrightarrow B, \quad b \mapsto (\xi_B^r \circ \omega_B)(1 \otimes_S b) = \sum b^{(1)} \varepsilon(b^{(2)})$$

is an antipode for  $B$ .

On one hand, we have

$$\begin{aligned}
& (\mu \circ \mathbf{a} \otimes_S B \circ \Delta)(b) \\
= & \sum \mathbf{a}(b_1)b_2 \\
= & \sum \mathbf{a}(b_1)b_2\varepsilon(b_3) \\
= & [\mu \circ B \otimes_S \xi_B^r](\sum \mathbf{a}(b_1) \otimes_S b_2 \otimes_S b_3) \\
= & [\mu \circ B \otimes_S \xi_B^r](\sum (b_1)^{(0)}\varepsilon((b_1)^{(1)}) \otimes_S b_2 \otimes_S b_3) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B](\sum (b_1)^{(0)} \otimes_S (b_1)^{(1)} \otimes_S b_2 \otimes_S b_3) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B \circ B \otimes_S B \otimes_S \Delta](\sum (b_1)^{(0)} \otimes_S (b_1)^{(1)} \otimes_S b_2) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B \circ B \otimes_S B \otimes_S \Delta \circ \omega_B \otimes_S B](\sum (1 \otimes_S b_1) \otimes_S b_2) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B \circ B \otimes_S B \otimes_S \Delta \circ \omega_B \otimes_S B \circ B \otimes_S \Delta](1 \otimes_S b) \\
\stackrel{(23)}{=} & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B \circ B \otimes_S B \otimes_S \Delta \circ B \otimes_S \Delta \circ \omega_B](1 \otimes_S b) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B \circ B \otimes_S B \otimes_S \Delta \circ B \otimes_S \Delta](\sum b^{(0)} \otimes_S b^{(1)}) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B \circ B \otimes_S B \otimes_S \Delta](\sum b^{(0)} \otimes_S (b^{(1)})_1 \otimes_S (b^{(1)})_2) \\
= & [\mu \circ B \otimes_S \xi_B^r \circ \xi_B^r \otimes_S B \otimes_S B](\sum b^{(0)} \otimes_S (b^{(1)})_1 \otimes_S (b^{(1)})_2 \otimes_S (b^{(1)})_3) \\
= & [\mu \circ B \otimes_S \xi_B^r](\sum b^{(0)}\varepsilon((b^{(1)})_1) \otimes_S (b^{(1)})_2 \otimes_S (b^{(1)})_3) \\
= & [\mu \circ B \otimes_S \xi_B^r](\sum b^{(0)} \otimes_S \varepsilon((b^{(1)})_1)(b^{(1)})_2 \otimes_S (b^{(1)})_3) \\
= & [\mu \circ B \otimes_S \xi_B^r](\sum b^{(0)} \otimes_S (b^{(1)})_1 \otimes_S (b^{(1)})_2) \\
= & \mu(\sum b^{(0)} \otimes_S (b^{(1)})_1 \varepsilon((b^{(1)})_2)) \\
= & \sum b^{(0)}(b^{(1)})_1 \varepsilon((b^{(1)})_2) \\
= & \xi_B^r(\sum b^{(0)}(b^{(1)})_1 \otimes_S (b^{(1)})_2) \\
= & (\xi_B^r \circ \gamma_B)(\sum b^{(0)} \otimes_S b^{(1)}) \\
= & (\xi_B^r \circ \gamma_B \circ \omega_B)(1 \otimes_S b) \\
= & \xi_B^r(1 \otimes_S b) \\
= & 1_B \varepsilon(b) \\
= & (\eta \circ \varepsilon)(b)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\mu \circ B \otimes_S \mathbf{a} \circ \Delta)(b) &= \sum b_1 \mathbf{a}(b_2) \\
&= \sum b_1 (b_2)^{(0)} \varepsilon((b_2)^{(1)}) \\
&= \xi_B^r(\sum b_1 (b_2)^{(0)} \otimes_S (b_2)^{(1)}) \\
&= \xi_B^r(\sum b_1 \omega_B(1 \otimes_S b_2)) \\
&= [\xi_B^r \circ \omega_B](\sum b_1 \otimes_S b_2) && (\omega_B \text{ is left } B\text{-linear}) \\
&= [\xi_B^r \circ \omega_B \circ \gamma_B](1 \otimes_S b) \\
&= \xi_B^r(1 \otimes_S b) && (\omega_B \circ \gamma_B = \text{id}_{B \otimes_S B}) \\
&= 1_B \varepsilon(b) \\
&= (\eta \circ \varepsilon)(b). \blacksquare
\end{aligned}$$

We present now the *Fundamental Theorem of Hopf Semimodules*.

**Theorem 4.13.** *The following are equivalent for an  $S$ -bisemialgebra  $(B, \mu, \eta, \Delta, \varepsilon)$  :*

1.  $B$  is a Hopf  $S$ -semialgebra;
2. For every  $M \in \mathbb{S}_B^B$ , we have an isomorphism in  $\mathbb{S}_B^B$  :

$$M^{\text{co}B} \otimes_S B \xrightarrow{\psi_M} M, \quad m \otimes_S b \mapsto mb; \quad (24)$$



3. For every  $M \in \mathbb{S}_B^B$ , we have an isomorphism in  $\mathbb{S}_B^B$  :

$$\mathrm{Hom}_B^B(B, M) \otimes_S B \xrightarrow{\varphi^M} M, f \otimes_S b \mapsto f(b); \quad (25)$$

4. We have an isomorphism in  $\mathbb{S}_B^B$  :

$$\mathrm{Hom}_B^B(B, B \otimes_S^a B) \otimes_S B \xrightarrow{\varphi_{B \otimes_S^a B}} B \otimes_S^a B. \quad (26)$$

**Proof.** (1)  $\Rightarrow$  (2) : Let  $\mathbf{a}$  be an antipode for  $H$ . It can be shown that  $\psi_M$  is an isomorphism with inverse given by  $m \mapsto \sum m_{<0>} \mathbf{a}(m_{<1>})$ .

(2)  $\Rightarrow$  (3) This follows from the isomorphisms

$$\mathrm{Hom}_B^B(B, M) \otimes_S B \xrightarrow[\text{(13)}]{v_M \otimes_S B} M^{\mathrm{co}B} \otimes_S B \xrightarrow{\psi_M} M$$

and the fact that  $\psi_M \circ v_M \otimes_S B = \varphi_M$  : for all  $f \in \mathrm{Hom}_B^B(B, M)$  and  $b \in B$  :

$$(\psi_M \circ (v_M \otimes_S B))(f \otimes_S b) = \psi_M(f(1_B) \otimes_S b) = f(1_B)b = f(b) = \varphi_M(f \otimes_S b).$$

(3)  $\Rightarrow$  (4) trivial.

(4)  $\Rightarrow$  (1) We have an isomorphism of  $S$ -semimodules

$$B \xrightarrow{\zeta_B} \mathrm{Hom}_B^B(B, B \otimes_S^a B), a \mapsto [b \mapsto \sum ab_1 \otimes_S^a b_2]$$

with inverse  $f \mapsto (\vartheta_B^r \circ B \otimes_S \varepsilon \circ f)(1_B)$ . Moreover, for all  $a, b \in B$  we have

$$(\varphi_{B \otimes_S B} \circ \zeta_B \otimes_S B)(a \otimes_S b) = \varphi_{B \otimes_S B}(\zeta_B(a) \otimes_S b) = \zeta_B(a)(b) = \sum ab_1 \otimes_S^a b_2 = \gamma_B(a \otimes_S b).$$

Consequently, we have an isomorphism in  $\mathbb{S}_B^B$  :

$$\gamma_B : B \otimes_S B \xrightarrow{\zeta_B \otimes_S B} \mathrm{Hom}_B^B(B, B \otimes_S^a B) \otimes_S B \xrightarrow{\varphi_{B \otimes_S B}} B \otimes_S^a B.$$

By Proposition 4.12,  $H$  is a Hopf  $S$ -semialgebra. ■

We are ready now to present the *Fundamental Theorem of Hopf Semialgebras*. Making use of Proposition 4.12, the proof is similar to that of [BW2003, 15.5].

**Theorem 4.14.** *The following are equivalent for an  $S$ -bisemialgebra  $B$  :*

1.  $B$  is a Hopf  $S$ -semialgebra;

2.  $\mathbb{S}_S \xrightarrow{- \otimes_S B} \mathbb{S}_B^B$  (an equivalence of categories) with inverse  $\mathrm{Hom}_B^B(B, -) : \mathbb{S}_B^B \longrightarrow \mathbb{S}_S$ .

**Proof.** (1)  $\Rightarrow$  (2) For every  $M_S$ , it is obvious that we have a natural isomorphism of  $S$ -semimodules

$$M \simeq \mathrm{Hom}_B^B(B, M \otimes_S B), m \mapsto [b \mapsto m \otimes_S b]$$

with inverse  $f \mapsto (\rho_M \circ f)(1_B)$ . It follows that  $\mathrm{Hom}_B^B(B, -) \circ - \otimes_S B \simeq \mathrm{id}_{\mathbb{S}_S}$ . On the other hand, we have a natural isomorphism  $\mathrm{Hom}_B^B(B, M) \otimes_S B \xrightarrow{\varphi^M} M$  in  $\mathbb{S}_B^B$ ; this means that  $- \otimes_S B \circ \mathrm{Hom}_B^B(B, -) \simeq \mathrm{id}_{\mathbb{S}_B^B}$ . Consequently,  $\mathbb{S}_B^B \xrightarrow{\mathrm{Hom}_B^B(B, -)} \mathbb{S}_S$  with inverse  $- \otimes_S B$ .

(2)  $\Rightarrow$  (1) This follows by Proposition 4.13. ■

*Remark 4.15.* We notice here that the Fundamental Theorem of Hopf Algebras can be obtained by applying results of [?] on *Hopf monads* taking into consideration that  $\mathbb{S}_S$  has colimits and that  $- \otimes_S B \simeq B \otimes_S - : \mathbb{S}_S \longrightarrow \mathbb{S}_S$  preserves colimits for every  $B_S$ . However, we follow in this paper the direct algebraic approach.

We present now the main reconstruction result in this Section:

**Theorem 4.16.** *Let  $H$  be an  $S$ -semimodule. There is a bijective correspondence between the structures of Hopf  $S$ -semialgebras on  $H$ , the Hopf monad structures on  $H \otimes_S - : \mathbb{S}_S \longrightarrow \mathbb{S}_S$  and the Hopf monad structures on  $- \otimes_S H : \mathbb{S}_S \longrightarrow \mathbb{S}_S$ .*

**Proof.** The proof of the bijective correspondence in (2) is similar to that of [Ver, Theorem 3.9] taking into consideration that  $\mathbb{S}_S$  is cocomplete, that  $S$  is a regular generator in  $\mathbb{S}_S$  and the fact that  $X \otimes_S - \simeq - \otimes_S X$  preserve colimits in  $\mathbb{S}_S$  for every  $S$ -semimodule  $X$ . ■

## Semisimple and Cosemisimple Hopf Semialgebras

**Definition 4.17.** Let  $\mathfrak{C}$  be a category. We say that an object  $V$  in  $\mathfrak{C}$  is *semisimple* (*simple*) iff every monomorphism  $\iota : U \longrightarrow V$  in  $\mathfrak{C}$  is a coretraction (an isomorphism). Moreover, we say that  $V$  is *completely reducible* iff  $V$  is a direct sum of simple objects in  $\mathfrak{C}$ .

*Remark 4.18.* In contrast with modules over a ring, a semisimple semimodule over a semiring is not necessarily completely reducible: an  $S$ -subsemimodule  $U \xrightarrow{\iota} V$  for which  $\iota$  is a coretraction is not necessarily a summand [Gol1999].

**Definition 4.19.** Let  $\mathfrak{C}$  and  $\mathfrak{D}$  be two categories such that  $\text{Obj}(\mathfrak{C}) \subseteq \text{Obj}(\mathfrak{D})$  and  $\text{Mor}_{\mathfrak{C}}(X, Y) \subseteq \text{Mor}_{\mathfrak{D}}(X, Y)$  for all  $X, Y \in \text{Obj}(\mathfrak{C})$ . We say that  $V \in \mathfrak{C}$  is  $(\mathfrak{C}, \mathfrak{D})$ -*semisimple* iff every monomorphism  $U \longrightarrow V$  in  $\mathfrak{C}$  which is a coretraction in  $\mathfrak{D}$  is also a coretraction in  $\mathfrak{C}$ .

**Definition 4.20.** Let  $H$  be a Hopf  $S$ -semialgebra.

1. We say that a left  $H$ -semimodule  $M$  is  
*semisimple* iff  $M$  is semisimple in  ${}_H\mathbb{S}$ ;  
 *$(H, S)$ -semisimple* iff  $M$  is  $({}_H\mathbb{S}, \mathbb{S}_S)$ -semisimple.
2. We say that a left  $H$ -secomodule  $N$  is  
*cosemisimple* iff  $N$  is semisimple in  ${}^H\mathbb{S}$ ;  
 *$(H, S)$ -cosemisimple* iff  $N$  is  $({}^H\mathbb{S}, \mathbb{S}_S)$ -semisimple.

**Definition 4.21.** We say that a Hopf  $S$ -semialgebra  $H$  is

*left semisimple* ( *$(H, S)$ -semisimple*) iff  $H \in {}_H\mathbb{S}$  is semisimple ( *$(H, S)$ -semisimple*);  
*left cosemisimple* ( *$(H, S)$ -cosemisimple*) iff  $H \in {}^H\mathbb{S}$  is cosemisimple ( *$(H, S)$ -cosemisimple*);  
*separable* iff  $H$  is a separable  $S$ -semialgebra (*i.e.* iff  $\mu_H : H \otimes_S H \longrightarrow H$  splits in  ${}_H\mathbb{S}_H$ );  
*coseparable* iff  $H$  is a coseparable  $S$ -semicoalgebra (*i.e.* iff  $\Delta_H : H \longrightarrow H \otimes_S H$  splits in  ${}^H\mathbb{S}^H$ ).

Symmetrically, one can introduce right-sided versions of these notions.

**Proposition 4.22.** *Let  $(H, \mu, \eta, \Delta, \varepsilon, a)$  be a Hopf  $S$ -semialgebra with  ${}_S H$  an  $\alpha$ -semimodule and consider  $H^{*rat} := \text{Rat}^H({}_H^* H^*)$ .*

1.  $B^{*rat} \in \mathbb{S}_H^H$  with right  $H$ -action given by  $(f \leftharpoonup h)(g) := f(a(h)g)$  for all  $f \in B^{*rat}$  and  $h, g \in H$ .

2. We have an isomorphism in  $\mathbb{S}_H^H$  :

$$\int^l H \otimes_S H \longrightarrow H^{*rat}, \quad t \otimes_S h \mapsto t \leftharpoonup h.$$

3.  $\int_l H = 0$  if and only if  $H^{*rat} = 0$ .

**Proof.** 1. Direct calculations [DNR2001, 5.2.1] show that  $H^{*rat} \in \mathbb{S}_H^H$ .

2. This follows directly from the fact that  $\int^l H = (H^{*rat})^{\text{co}H}$  (Lemma 2.22) and the Fundamental Theorem of Hopf Semimodules (Proposition 4.13) applied to  $M = H^{*rat}$ .

3. If  $H^{*rat} = 0$ , then  $\int^l H = (H^{*rat})^{\text{co}H} = 0$ . On the other hand, if  $\int^l H = 0$ , then  $H^{*rat} \simeq \int^l H \otimes_S H = 0 \otimes_S H = 0$ . ■

The following result characterizes coseparable Hopf  $S$ -semialgebras.

**Proposition 4.23.** *The following are equivalent for a Hopf  $S$ -semialgebra  $H$  :*

1.  $\int^{l,1} H \neq \emptyset$ ;
2. Every left (right)  $H$ -semicomodule is  $(H, S)$ -cosemisimple;
3.  $H$  is left (right)  $(H, S)$ -cosemisimple;
4.  $H$  is coseparable;
5. There exists an  $S$ -linear map  $\delta : H \otimes_S H \longrightarrow S$  such that

$$\delta \circ \Delta_H = \varepsilon \text{ and } (H \otimes_S \delta) \circ (\Delta_H \otimes_S H) = (\delta \otimes_S H) \circ (H \otimes_S \Delta_H);$$

6.  $\int^{r,1} H \neq \emptyset$ .

**Proof.** The proof can be obtained by arguments similar to a combination of [BW2003, 16.10] and [BW2003, 16.10] (see also [Abe1980, Theorem 3.3.2]). ■

**Corollary 4.24.** *Let  $H$  be a Hopf  $S$ -semialgebra. If  ${}_S H$  is an  $\alpha$ -semimodule, then the following are equivalent:*

1.  $H$  is left  $(H, S)$ -cosemisimple;
2.  $H$  is a semisimple right  $H^*$ -semimodule;
3. every rational right  $H^*$ -semimodule is semisimple.

**Proof.** If  ${}_S H$  is an  $\alpha$ -semimodule, then we have an isomorphism of categories  ${}^H \mathbb{S} \simeq {}^H \text{Rat}(\mathbb{S}_{H^*})$  and the result follows by Proposition 4.23. ■

*Example 4.25.* Let  $S$  be such that every  $S$ -subsemimodule of  $H$  is injective. If  $H$  has a total left integral, then  $H$  is left cosemisimple by Proposition 4.23.

The following result characterizes separable Hopf  $S$ -semialgebras.

**Proposition 4.26.** *The following are equivalent for a Hopf  $S$ -semialgebra  $H$  :*

1.  $\int_{l,1} H \neq \emptyset$ ;
2. Every left (right)  $H$ -semimodule is  $(H, S)$ -semisimple;
3.  $H$  is left (right)  $(H, S)$ -semisimple;
4.  $H$  is separable;
5. There exists  $e = \sum e^1 \otimes_S e^2 \in H \otimes_S H$  such that

$$he = eh \text{ for all } h \in H \text{ and } \sum e^1 e^2 = 1_H;$$

6.  $\int_{r,1} H \neq \emptyset$ .

**Proof.** The proof is similar to that of [BW2003, 16.13] (see also [CMZ2002]). ■

## 5 Dual Bisemialgebras and Dual Hopf

As before,  $S$  denotes a commutative semiring with  $1_S \neq 0_S$  and  $\mathbb{S}_S$  is the category of  $S$ -semimodules.

**Lemma 5.1.** *Let  $X, Y$  be  $S$ -semimodules. If  $X_S$  is finitely generated and projective and  $Y_S$  is uniformly finitely presented, then we have a canonical isomorphism*

$$X^* \otimes_S Y^* \xrightarrow{\mathfrak{h}_{(X,Y)}} (X \otimes_S Y)^*, \quad f \otimes_S g \mapsto [x \otimes_S y \mapsto f(x)g(y)]. \quad (27)$$

**Proof.** Since  $X_S$  is finitely generated and projective,  $X_S^*$  is also (finitely generated and) projective whence flat. The given map is the composition of the following isomorphisms

$$X^* \otimes_S \text{Hom}_S(Y, S) \xrightarrow{v_{(X^*, Y, S)}} \text{Hom}_S(X, \text{Hom}_S(Y, S) \otimes_S S) \simeq \text{Hom}_S(X \otimes_S Y, S).$$

where  $v_{(X^*, Y, S)}$  is an isomorphism by Lemma 1.12 and the second isomorphism is the canonical one indicating the adjointness of the tensor and hom functors. ■

**Proposition 5.2.** *Let  $A$  be an  $S$ -semimodule with  $A_S$  uniformly finitely presented projective and consider the isomorphism of semirings  $S^* \xrightarrow{h} S$ .*

1. *If  $(A, \mu, \eta)$  is an  $S$ -semialgebra, then  $(A^*, \mathfrak{h}_{(A,A)}^{-1} \circ \mu^*, h \circ \eta^*)$  is an  $S$ -semicoalgebra.*
2. *If  $(A, \mu, \eta, \Delta, \varepsilon)$  is an  $S$ -bisemialgebra, then  $(A^*, \Delta^* \circ \mathfrak{h}_{(A,A)}, \varepsilon^* \circ h^{-1}, \mathfrak{h}_{(A,A)}^{-1} \circ \mu^*, h \circ \eta^*)$ , is an  $S$ -bisemialgebra.*
3. *If  $(A, \mu, \eta, \Delta, \varepsilon, \mathfrak{a})$  is a Hopf  $S$ -semialgebra, then  $(A^*, \Delta^* \circ \mathfrak{h}_{(A,A)}, \varepsilon^* \circ h^{-1}, \mathfrak{h}_{(A,A)}^{-1} \circ \mu^*, h \circ \eta^*, \mathfrak{a}^*)$  is a Hopf  $S$ -semialgebra.*

**Proof.** Applying Lemma 5.1 twice, we have the canonical isomorphisms

$$(A \otimes_S A)^* \xrightarrow{\mathfrak{h}_{(A,A)}} A^* \otimes_S A^* \text{ and } (A \otimes_S A \otimes_S A)^* \xrightarrow{\mathfrak{h}_{(A \otimes_S A, A)}} (A \otimes_S A)^* \otimes_S A^* \xrightarrow{\mathfrak{h}_{(A,A)} \otimes_S A^*} A^* \otimes_S A^* \otimes_S A^*.$$

The result follows now by applying  $(-)^* := \text{Hom}_S(-, S)$  to the  $S$ -semimodules and arrows defining the given  $S$ -semialgebra (resp.  $S$ -bisemialgebra, Hopf  $S$ -semialgebra).

## The finite dual

**Definition 5.3.** We say that an  $S$ -semimodule  $M$  is Noetherian iff every  $S$ -subsemimodule of  $M$  is finitely generated. We say that the semiring  $S$  is right (left) Noetherian iff  $S_S$  ( ${}_S S$ ) is Noetherian. If  $S$  is right and left Noetherian, then we say that  $S$  is Noetherian.

**Definition 5.4.** We say that the semiring  $S$  is *right (left)  $m$ -Noetherian* iff every finitely generated right (left)  $S$ -semimodule is Noetherian. If  $S$  is right and left  $m$ -Noetherian, then we say that  $S$  is an  *$m$ -Noetherian semiring*.

*Remark 5.5.* The notion of  $m$ -Noetherian semirings was considered in [EM2011] where such semirings were called Noetherian. To avoid confusion, we add the prefix  $m$  which refers to modules.

*Example 5.6.* Every *finite* semiring is  $m$ -Noetherian. Such rings are not rare: Let  $n$  be a positive integer and  $X_n = \{-\infty, 0, 1, \dots, n\}$ . Define addition and multiplication on  $X_n$  as

$$i +_{X_n} h := \max\{i, h\} \text{ and } i \cdot_{X_n} h = \min\{i + h, n\}.$$

One can easily check that  $\{X_n, +_{X_n}, \cdot_{X_n}, \mathbf{0}, \mathbf{1}\}$  is a (finite) semiring with  $\mathbf{0} = -\infty$  and  $\mathbf{1} = 0$ . Moreover,  $X_n$  has no non-zero zerodivisors and is additively idempotent since

$$\mathbf{1} +_{X_n} \mathbf{1} = 0 +_{X_n} 0 = \max\{0, 0\} = 0 = \mathbf{1}.$$

*Example 5.7.* ([BMS2013, Example 2.2]) The semiring  $(\mathbb{N}_0, +, \cdot)$  is Noetherian but not  $m$ -Noetherian: the semimodule  $\mathbb{N}_0 \times \mathbb{N}_0$  is finitely generated but its subsemimodule generated by  $\{(n+1, n) \mid n \in \mathbb{N}\}$  is not finitely generated.

**Lemma 5.8.** ([BMS2013, Proposition 2.5]) *A semiring  $S$  is  $m$ -Noetherian if and only if every  $S$ -subsemimodule of a finitely generated free  $S$ -semimodule  $S^n$  is finitely generated.*

**Lemma 5.9.** *Let  $S$  be an  $m$ -Noetherian semiring.*

1. *Every finitely generated  $S$ -semimodule is finitely presentable.*
2. *Every uniformly finitely generated  $S$ -semimodule is uniformly finitely presented.*

**Proof.** 1. This is [BMS2013, Proposition 2.6].

2. This follows directly from the definitions. ■

**Lemma 5.10.** *Let  $\Lambda$  and  $\Lambda'$  be two index sets. If  $S$  is an  $m$ -Noetherian semiring, then the following canonical map is injective*

$$\beta_{(\Lambda, \Lambda')} : S^\Lambda \otimes_S S^{\Lambda'} \longrightarrow S^{\Lambda \times \Lambda'}, \quad f \otimes_S f' \mapsto [(\lambda, \lambda') \mapsto f(\lambda)f'(\lambda')].$$

**Proof.** Write  $S^\Lambda = \varinjlim M_\lambda$ , a direct limit of *uniformly* finitely generated  $S$ -subsemimodules. Since  $S$  is  $m$ -Noetherian,  $M_\lambda$  is uniformly finitely presented for each  $\lambda \in \Lambda$ . By Lemma 1.10, we have for each  $\lambda \in \Lambda$  an isomorphism of  $S$ -semimodules  $M_\lambda \otimes_S S^{\Lambda'} \simeq M_\lambda^{\Lambda'}$ . It follows that

$$S^\Lambda \otimes_S S^{\Lambda'} = \varinjlim M_\lambda \otimes_S S^{\Lambda'} \simeq \varinjlim (M_\lambda \otimes_S S^{\Lambda'}) \simeq \varinjlim M_\lambda^{\Lambda'} \hookrightarrow (S^\Lambda)^{\Lambda'} \simeq S^{\Lambda \times \Lambda'}. \blacksquare$$

**5.11.** Let  $(A, \mu, \eta)$  be an  $S$ -semialgebra such that the canonical morphism of  $(A, A)$ -bisemimodules

$$\beta_{(A,A)} : R^A \otimes_S R^A \hookrightarrow R^{A \times A}, \quad f \otimes_S f' \mapsto [(a, a') \mapsto f(a)f'(a')] \quad (28)$$

is injective. Notice that the multiplication  $\mu : A \otimes_S A \longrightarrow A$  induces an (injective)  $(A, A)$ -bilinear map

$$\mu^\bullet : S^A \longrightarrow S^{A \times A}.$$

We define  $A^\circ$  as the pullback in the following diagram in  ${}_A\mathbb{S}_A$  :

$$\begin{array}{ccc} A^\circ & \xrightarrow{\quad \quad} & S^A \otimes_S S^A \\ \downarrow & & \downarrow \beta_A \\ A^* & \xrightarrow{\quad \mu^\bullet \quad} & S^{A \times A} \end{array}$$

equivalently

$$\begin{aligned} A^\circ &= \{ (h, \sum_{i=1}^n f_i \otimes_S g_i) \in A^* \oplus (S^A \otimes_S S^A) \mid \mu^\bullet(h) = \beta_{(A,A)}(\sum_{i=1}^n f_i \otimes_S g_i) \} \\ &\simeq \{ h \in A^* \mid \exists \{(f_i, g_i)\}_{i=1}^n \subseteq S^A \otimes_S S^A \text{ with } h(ab) = \sum_{i=1}^n f_i(a)g_i(b) \text{ for all } a, b \in A \}. \end{aligned}$$

**Lemma 5.12.** *Let  $A$  be an  $S$ -semialgebra and consider  $A^*$  as an  $(A, A)$ -bisemimodule in the canonical way. If  $S$  is  $m$ -Noetherian, then the following are equivalent for  $f \in A^*$  :*

1.  $f \in A^\circ$ ;
2.  $Af$  is a finitely generated  $S$ -semimodule;
3.  $\mu^\bullet(f) \in \beta_{(A,A)}(A^\circ \otimes_S S^A)$ ;
4.  $\mu^\bullet(f) \in \beta_{(A,A)}(A^* \otimes_S S^A)$ ;
5.  $fA$  is finitely generated;
6.  $\mu^\bullet(f) \in \beta_{(A,A)}(S^A \otimes_S A^\circ)$ ;
7.  $\mu^\bullet(f) \in \beta_{(A,A)}(S^A \otimes_S A^*)$ ;
8.  $\mu^\bullet(f) \in \beta_{(A,A)}(A^\circ \otimes_S S^A) \cap \beta_{(A,A)}(S^A \otimes_S A^\circ)$ .

**Proof.** Notice that  $A^\circ \hookrightarrow A^*$  is an  $(A, A)$ -subbisemimodule since it is – by definition – a pullback in  ${}_A\mathbb{S}_A$ . The rest of the technical proof is now similar to that of [AG-TW2000, Proposition 1.6]. ■

**Theorem 5.13.** *Let  $S$  be  $m$ -Noetherian and  $A$  an  $S$ -semialgebra. If  $A^\circ \hookrightarrow S^A$  is pure and  $S^A$  is mono-flat, then  $A^\circ$  is an  $S$ -semicoalgebra.*

**Proof.** Since  $S$  is  $m$ -Noetherian, the following  $S$ -linear map

$$\beta_{(A,A)} : S^A \otimes_S S^A \longrightarrow S^{A \times A}, \quad f \otimes_S f' \mapsto [(a, a') \mapsto f(a)f'(a')]$$

is injective by Lemma 5.10. Since  $A^\circ \hookrightarrow S^A$  is pure and  $S^A$  is mono-flat, we have the following canonical embeddings

$$A^\circ \otimes_S A^\circ \hookrightarrow S^A \otimes_S A^\circ \hookrightarrow S^A \otimes_S S^A \xrightarrow{\beta_{(A,A)}} S^{A \times A}$$

and the following canonical map

$$\tilde{\beta}_A : A^\circ \otimes_S (A^\circ \otimes_S A^\circ) \hookrightarrow S^A \otimes_S (A^\circ \otimes_S A^\circ) \hookrightarrow S^A \otimes_S S^{A \times A} \xrightarrow{\beta_{(A, A \times A)}} S^{A \times A \times A}.$$

Moreover, for every  $f \in A^\circ$ , we have by Lemma 5.12:

$$\mu^*(f) \in \beta_{(A,A)}(A^\circ \otimes_S S^A) \cap \beta_{(A,A)}(S^A \otimes_S A^\circ) = \beta_{(A,A)}(A^\circ \otimes_S S^A \cap S^A \otimes_S A^\circ) = \beta_{(A,A)}(A^\circ \otimes_S A^\circ).$$

Consider the following diagram

$$\begin{array}{ccccc}
S^A & \xrightarrow{\mu^\bullet} & & & S^{A \times A} \\
\downarrow \mu^\bullet & \swarrow & A^\circ \xrightarrow{\Delta} A^\circ \otimes_S A^\circ & \searrow & \downarrow (id \times \mu)^\bullet \\
& & \Delta \downarrow & & \\
& & A^\circ \otimes_S A^\circ & \xrightarrow{\Delta \otimes_S id} & A^\circ \otimes_S A^\circ \otimes_S A^\circ \\
& \swarrow & & \searrow & \\
S^{A \times A} & \xrightarrow{(\mu \times id)^\bullet} & & & S^{A \times A \times A}
\end{array}$$

$\beta_{(\tilde{A}, A)}$

with

$$\mu^\bullet : S^A \longrightarrow S^{A \times A}, \quad f \mapsto [(a, b) \mapsto f(ab)] \text{ and } \Delta = \mu_{|A^\circ}^\bullet : A^\circ \longrightarrow A^\circ \otimes_S A^\circ.$$

Since the multiplication  $\mu_A$  is associative, the outer rectangle is clearly commutative. Moreover, all trapezoids are commutative. Since the canonical map  $\tilde{\beta}_A$  is injective, we conclude that the inner diagram is commutative, *i.e.*  $\Delta$  is coassociative. It is not difficult to show that the restriction of  $\eta^* : A^* \longrightarrow S^* \simeq S$  is a counity for  $A^\circ$ . ■

**Theorem 5.14.** *Let  $S$  be  $m$ -Noetherian.*

1. *If  $B$  is an  $S$ -bisemialgebra with  $B^\circ \hookrightarrow S^B$  pure and  $S^B$  mono-flat, then  $B^\circ$  is an  $S$ -bisemialgebra.*
2. *If  $H$  is a Hopf  $S$ -semialgebra with  $H^\circ \hookrightarrow S^H$  pure and  $S^H$  mono-flat, then  $H^\circ$  is a Hopf  $S$ -semialgebra.*

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